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$k=\bar{k}$, H a f.d. Hopf algebra $/k$, A associative unital algebra$/k$

Def. A right coaction of H on A is a right H-comodule structure $\rho: A \to A\otimes H$ which is an algebra homomorphism.

Dual notion: action of H on A

Def. A left action of H on A is a left H-module structure on A s.t. $h(ab)=h_1(a)h_2(b)$, $h(1)=\varepsilon(h)1$.

An H-action = an H^*-coaction = finite quantum symmetry

Def. Action of H on a module V is inner faithful if it does not factor through a smaller Hopf algebra (= coaction is inner faithful if it comes from one by a proper Hopf subalgebra). Can always replace an (co)action by an inner faithful one.

Ex. Group actions: if a group G acts on A then kG acts on A, $|G|<\infty$ $\mathrm{Fun}(G,k)$ coacts on A. ("classical symmetry").

Main question: for which A does there exist an action not factoring through a group algebra? $\Leftrightarrow$ inner faithful action which is not a group action? If not, say no fin. quantum symmetry.

Ex. V a repr of $H \Rightarrow H$ acts on TV (inner faithful if V is inner faithful). Also H acts on $\mathrm{End}\,V$ by conjugation.

So restrict to the case when A is a domain with special properties.

Thm. (Skryabin) $HGA \Rightarrow H \curvearrowright Q_A$

(1.) A a commutative domain.

<u>Thm.</u> (E-Walton) If H is semisimple + cosemisimple then any H-action on A factors through a group algebra. (no quantum symmetry).

<u>Pf</u> (Idea). Enough A f.gen. $X = \operatorname{Spec} A$. $\forall \chi \in X$
$\chi : A \to k$

$\rho_\chi : A \to H^*$, $\operatorname{Im} \rho_\chi = A_\chi$ coideal subalgebra.
$\mapsto (\chi \otimes 1)(\rho)$

<u>Schneider</u> : H s.s. + cos.s. $\Rightarrow$ finitely many coideal subalgebras $\Rightarrow$ for $\chi \in U \subset X$ open $\neq \emptyset$
$A_\chi = B$ fixed $\Rightarrow$ B a Hopf subalgebra and
$\rho : A \to A \otimes B$, so B commutative and
H-coaction factors through B^*-cocommutative + cosemisimple
$=$ group algebra.

<u>Thm.</u> (Skryabin) H ss <u>or</u> coss $\Rightarrow$ has finitely many coideal subalgebras.

<u>Cor.</u> H coss $\Rightarrow$ action factors through gp algebra
H ss $\Rightarrow$ factors through ~~cocomm~~ s.s. group scheme.

<u>Nonsemisimple case</u>: many examples.
e.g. $H = \langle g, x \rangle$ Sweedler, $A = \mathbb{R}[z]$,
$gz = -z$, $xz = 1$.

But all not Hopf-Galois $\left(\begin{array}{c} HG \Leftrightarrow \\ A \otimes_{A^H} A \to A \otimes H^* \\ \text{an isom} \end{array} \right)$

H Galois-theoretical if can act inner faithf. on a field. Not known which HA are GT.

<u>Ex.</u> $u_q(sl_2)$ is GT but $\operatorname{gr} u_q(sl_2)$ is not.

(2.) A noncommutative. We consider the case when A is a division algebra. Assume char $k = 0$
(if A is Ore and $H \circlearrowright A \Rightarrow H \circlearrowright Q_A$). Skryabin - van Oystayen.

<u>Q.</u> Can any f.d. Hopf alg act inner faithfully on a div. algebra?

<u>Conj.</u> (Artamonov) Any f.d. Hopf algebra can act on a quantum torus, $x_i x_j = q_{ij} x_j x_i$.

<u>Question.</u> Is this true already if q_{ij} are roots of unity? Is it at least true if A is a central ~~..~~ division algebra?

<u>Special case</u>: $H = \mathrm{Fun}(G, k)$. Then an H-action is simply a G-grading on A, and an inner faithful action = faithful grading ($A_g \neq 0$ $\forall g$).

<u>Q.</u> For which G $\exists$ such on a central div. algebra A of degree d?

<u>Thm.</u> (CE) $\exists$ such grading $\Leftrightarrow$ G has a normal abelian subgroup of index dividing d

<u>Q.</u> If H is semisimple and acts inner faithfully on a (central) division algebra A, does the action have to be Hopf-Galois (yes in the commutative case).

<u>Q.</u> Which semisimple Hopf algebras can act inner faithfully on (central) div. algebras?

<u>Prop.</u> Twisted group algebras $\mathbb{C}[G]^J$ can act.
(if V is a faithful repr., $A = Q_{(S(V))^J}$ - this is a domain)

<u>Thm</u> (CE). If $H \curvearrowright A \leftarrow$ central div. alg of degree d, action Hopf-Galois $\Rightarrow$ $\mathrm{PIdeg}(H^*) \leq d^2$ (in partic, all irreps of H^* have dim $\leq d^2$).
This is sharp (example of $=$ comes from $\mathbb{C}G^J$).

(3.) $A =$ Weyl algebra, char $k = 0$.

<u>Thm.</u> $H \curvearrowright A \Rightarrow$ action factors through a group algebra.

<u>Proof:</u> Let D be a division algebra of degree d, $H \curvearrowright D$.

<u>Key lemma.</u> If $(\dim(H)!, d) = 1$ then $H \curvearrowright Z(D)$. If $H \curvearrowright D$ inner faithfully $\Rightarrow$ $H \curvearrowright Z$ inn. faith.

<u>Pf.</u> $[D : D^H] \leq \dim H$ by thm of Cohen-Fischman-Montgomery, so $\deg(D^H) = d$
$\Rightarrow D = Z \cdot D^H \Rightarrow Z = C(D^H) \Rightarrow H \curvearrowright Z$.

[The second statement follows from $D = Z \cdot D^H$

Now thm follows for semisimple H from the result of EW, as follows.

Reduce mod p: $H \curvearrowright A \Rightarrow H_p \curvearrowright A_p$ (over $\mathbb{F}_p$)
action inner faithful for large p.

$\Rightarrow H_p \curvearrowright Q_{A_p}$ ← central div. algebra of degree p^{2n}. So if $p >\dim H$, the key lemma applies. Thus, H_p acts inner faithfully on Z_p. But for large p H_p is semisimple and cosemisimple. $\Rightarrow$ By EW, H_p is cocommutative $\Rightarrow H$ is cocommutative $\Rightarrow H$ is a group algebra.

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④ Generalization to H nonsemisimple:

Can't use EW any more.

So use the following thm:

~~Thm~~ Z a field of char p / k, $H \curvearrowright Z$

Def. Action is Frobenius if $H \curvearrowright Z^{p^i}$ $\forall i$.

Ex. H = Sweedler, $Z = k(z)$, p odd

$gz = -z$, $xz = 1$ not Frobenius since

$xz^2 = 0$, so $xz^p = z^{p-1} \notin Z^p$.

Prop. $H \curvearrowright Z$ Frobenius $\Rightarrow$ factors through a group algebra. $\bigcap Z^{p^i} = k$

Pf. Prove Z_x indep of x using Frobenius condition. (Z_x defined over $\bigcap Z^{p^i} = k$)

To use this theorem, we have to prove that $H \curvearrowright Z^{p^i}$ $\forall i$ (otherwise proof is the same)

To do so, we need to consider reduction not just mod p, but mod p^m for all m, getting algebras over the truncated Witt ring of $\overline{\mathbb{F}}_p$, $W_m(\overline{\mathbb{F}}_p)$ (an algebra over $\mathbb{Z}/p^m\mathbb{Z}$). Namely, we have to show that H preserves the center $Z(m)$ of the reduction $Q(A_{p^m})$. This center ~~is generated by~~ (full localiz) ~~$x_i^{p^m}, y_i^{p^m}$~~ modulo p is $Z^{p^{m-1}}$ (generated by $x_i^{p^m}, y_i^{p^m}$), so this suffices.

To show $H \circlearrowright Z(m)$, it suffices to show that $Z(m) = C(Q(A_{p^m})^H)$. Clearly $Z(m) \subset C(Q(A_{p^m})^H)$, so need to show that $C(Q(A_{p^m})^H)$ is "large". For this show that any invariant mod p lifts mod p^m. It turns out that for this it suffices to show that there are "sufficiently many" invariants mod p^m. But these can be obtained by reducing mod p^m of invariants in $Q(A)$ in char 0, which are many by Cohen–Fishman–Montgomery.

<u>Subtelty</u>: <u>Don't</u> reduce Q_A mod p^m, just some elements of it.

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⑤ What are possible generalizations?

A a filtered quantization of a f.g. commutative domain, $\mathrm{gr} A = A_0$.

Then can reduce A, A_0 mod sufficiently large prime p s.t. $\mathrm{gr}(A_p) = A_{0p}$, and A_{0p} is a f.g. domain.

<u>Def</u>. A is residually PI if A_{0p} is PI for $p \gg 0$. E.g. if $A_0 = k[x_1, \dots, x_n]$ $\deg x_i = d_i$ ("nonlinear Lie algebra").

<u>Question</u>. Are all A residually PI?

This holds in all known examples.

<u>Thm</u>. Let B be a filtered quantization of B_0 in char p, B_0 comm. f.g. domain. Then if ~~PI~~ B is PI then $\mathrm{PI}\deg(B) = p^m$ for some m.

Thus we have:

Thm. If A is a residually PI quantization of a comm. f.g. domain A_0 over k char $k = 0$ then any action of a semisimple H on A factors through a group action.

Examples: 1. Twisted Differential operators on an affine variety.

2. Finite W-algebra + quot. ~~mod~~ mod central character

3. Quantized quiver varieties

4. Spherical symplectic refl. alg.

5. Enveloping alg of a f.dim Lie alg.

6. Tensor products of these with a comm. domain.

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(5) When can we generalize to ~~noncommutati~~ nonsemisimple setting?

Need ~~to~~ the property that $C(Q(A_{p^m})^H)/_p = Z^{p^{m-1}}$. For this we need some nondegeneracy condition.

Prop. This works if Spec B_0 is a generically symplectic Poisson variety (has an open symplectic leaf). So holds in most of the examples above. (except $U(\mathfrak{g})$, W-algebras).

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⑦ Can we generalize this to non-filtered quantizations, s.t. quantum tori?

Not immediately, since we have issues with the condition $(\dim H!, d) = 1$. Namely, e.g. take $xy = qyx$. If q is transcendental, no problem, and can in fact remain in char 0 (can specialize q to a root of unity of degree $> \dim H$ and run the original proof).

for nonsemisimple need to reduce mod p^m.

But for algebraic q have an issue: the order of q mod p is <u>not</u> a power of p, can have small prime factors!

Way out: a lemma in ANT.

Lemma. Let $q \in \bar{\mathbb{Q}}$ be not a root of 1. Then $\forall N$ $\exists$ a positive Dirichlet density of primes s.t. order of q mod p is coprime to N.

Now can run our proof for $N = \dim H$; it suffices to reduce mod (powers) of ∞ many primes not necessarily all sufficiently large primes. So get

Thm. Let $q \neq 0$ be not a root of 1. If a f.d. Hopf algebra H acts on $A = \langle xy = qyx \rangle$ then action factors through a group algebra.

What about higher dimensional quantum tori? Quantum homog. coordinate rings of abelian varieties? Elliptic algebras (in particular, Sklyanin algebras).

Need a generalization of the above NT lemma. This is where a result of A. Perrucca (2009) comes handy.

Let G be a torus or an abelian variety over a number field K. Let $\sigma \in G(K)$. Let G_σ be the Zariski closure of $\langle \sigma \rangle$. Let $(d, |G_\sigma/G_\sigma^\circ|) = 1$

<u>Thm.</u> (Perrucca). $\exists$ a positive Dirichlet density of primes in K mod which σ has order coprime to ~~[illegible]~~ d.

Now can use it to generalize our thm. Let $q = (q_{ij})_{1 \le i, j \le n}$, $q_{ij} q_{ji} = 1$, and $A_q = \langle x_i \mid x_i x_j = q_{ij} x_j x_i \rangle$. Then take $G = (k^\times)^{n(n-1)/2}$, $G_q \subset G$, $G_q = \langle q \rangle$.

<u>Thm.</u> If $|G_q/G_q^\circ|$ is coprime to $\dim H!$ then any action of a semisimple H on A_q factors through a gp. action.

Computation of $|G_q/G_q^\circ|$:

~~$|G_q/G_q^\circ| = |\{\chi(q) \mid \chi : G \to k^\times\}|$, $\chi(g)$ is a root of 1~~

~~i.e. $|G_q/G_q^\circ| = n \iff$ this group is μ_n.~~

$|G_q/G_q^\circ|$ ~~[illegible]~~ = maximal finite order of $\chi(q)$ for $\chi: G \to k^\times$

Nonss generalization: need nondg. condition. q is nondegenerate if $q(?, a)$ a nontriv char $\mathbb{Z}^n \to k^\times$ if $a \neq 0$. (we view q as a skewsymm. bicharacter $\mathbb{Z}^n \times \mathbb{Z}^n \to k^\times$).

<u>Thm.</u> If q nondg, $(\dim H!, |G_q/G_q^\circ|) = 1$ then any action of H on A_q factors through a group algebra.

(8) <u>Twisted homog. coord. rings:</u>

X abelian variety, $\mathcal{L}$ ample line bdle, char $k = 0$, $\sigma \in X$

$$B(X, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}})$$

(Artin - Tate - Van den Bergh).

<u>Theorem.</u> $H \curvearrowright B(X, \mathcal{L}, \sigma)$ semisimple Hopf action, $(\dim H!, |G_\sigma/G_\sigma^\circ|) = 1$ $\Rightarrow$ factors through a group algebra.

X elliptic curve, $|\sigma| = \infty$

$\Rightarrow$ true without semisimplicity condition.

Similar results for sklyanin algebras.

(9.) Deformation quantizations:

A def. quantization $/k[[\hbar]]$ of a commutative domain A_0/k, char $k=0$

i.e. $A \cong A_0[[\hbar]]$ as a $k[[\hbar]]$-mod.,

$A/\hbar A \cong A_0$ as an algebra.

$H \curvearrowright A$. H s.s. $\Rightarrow$ factors through a group algebra since it's so mod $\hbar$ by EW. What if H is not s.s.?

When can nontrivial actions be quantized?

<u>Thm.</u> If the Poisson center of $Q(A_0)$ is trivial, action factors through a group action.