Computing of Cocharacter Sequences of PI-algebras – 2

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Let K be a field of characteristic 0. We consider associative unital K-algebras R only. Let $K\langle X\rangle = K\langle x_1, x_2, \ldots\rangle$ be the free associative algebra. For a PI-algebra R with T-ideal $T(R) \subset K\langle X\rangle = K\langle x_1, x_2, \ldots\rangle$ let

$$\chi_n(R) = \sum_{\lambda \vdash n} m_\lambda(R)\chi_\lambda, \quad n = 0, 1, 2, \ldots,$$

be the cocharacter sequence of R.

This is the sequence of characters of the S_n-modules

$$P_n(R) = P_n/(P_n \cap T(R)), \quad n = 0, 1, 2, \dots,$$

where P_n is the space of multilinear polynomials of degree n in $K\langle X\rangle$. Here χ_λ is the irreducible character of S_n corresponding to the partition

$$\lambda = (\lambda_1, \dots, \lambda_n) \vdash n, \quad \lambda_1 \geq \cdots \geq \lambda_n \geq 0, \quad \lambda_1 + \cdots + \lambda_n = n,$$

and the nonnegative integers $m_\lambda(R)$ are the multiplicities of χ_λ in $\chi_n(R)$.

Our approach to find $m_\lambda(R)$ is to calculate the Hilbert (or Poincaré) series

$$H(F_d(R), T_d) = H(F_d(R), t_1, \ldots, t_d)$$

$$= \sum_{n_i \geq 0} \dim F_d(R)^{(n_1,\ldots,n_d)} t_1^{n_1} \cdots t_d^{n_d},$$

where $F_d(R)^{(n_1,\ldots,n_d)}$ is the multihomogeneous component of degree $(n_1, \ldots, n_d)$ of the relatively free algebra $F_d(R) = K\langle x_1, \ldots, x_d\rangle/(T(R) \cap K\langle x_1, \ldots, x_d\rangle)$.

This Hilbert series is a symmetric function which decomposes as a series of Schur functions

$$H(F_d(R), T_d) = \sum m_\lambda(R) S_\lambda(T_d)$$

and for $\lambda = (\lambda_1, \dots, \lambda_d)$ the multiplicities $m_\lambda(R)$ are the same as in the cocharacter sequence of R.

We consider the generating function of the multiplicities

$$M(R, T_d) = M(H(F_d(R)), T_d) = \sum m_\lambda(R) T_d^\lambda$$

$$= \sum m_\lambda(R) t_1^{\lambda_1} \cdots t_d^{\lambda_d}$$

which we call the multiplicity series of R.

If

$$\mathbb{C}[[T_d]]_\geq = \{ \sum_{n_1 \geq \cdots \geq n_d} a_n T_d^n = \sum_{n_1 \geq \cdots \geq n_d} a_n t_1^{n_1} \cdots t_d^{n_d} \} \subset \mathbb{C}[[T_d]],$$

then every element of $\mathbb{C}[[T_d]]_\geq$ is a multiplicity series of a symmetric function in $\mathbb{C}[[T_d]]^{S_d}$.

If we know the multiplicity series and if we can expand it as a series we shall know the multiplicities.

Theorem of Belov

For a PI-algebra R the Hilbert series of $F_d(R)$ is of the form

$$H(F_d(R), T_d) = \frac{p(T_d)}{\prod(1 - T^a)^{b_a}} = \frac{p(T_d)}{\prod(1 - t_1^{a_1} \cdots t_d^{a_d})^{b_a}},$$

where $p(T_d)$ is a polynomial, $a_i \geq 0$, $b_a > 0$.

Berele calls such functions nice rational functions.

Theorem of Berele

For a PI-algebra R the multiplicity series $M(R, T_d)$ is a nice rational function.

There are several ways to define Schur functions. The one that fits to our considerations is to define them as fractions of Vandermonde type determinants

$$S_\lambda(T_d) = \frac{V(\lambda+\delta, T_d)}{V(\delta, T_d)},$$

where $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\delta = (d-1, d-2, \ldots, 2, 1, 0)$, and for $\mu = (\mu_1, \ldots, \mu_d)$

$$V(\mu, T_d) = \begin{vmatrix} t_1^{\mu_1} & t_2^{\mu_1} & \cdots & t_d^{\mu_1} \\ t_1^{\mu_2} & t_2^{\mu_2} & \cdots & t_d^{\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{\mu_d} & t_2^{\mu_d} & \cdots & t_d^{\mu_d} \end{vmatrix}.$$

Lemma

Let $f(T_d) \in \mathbb{C}[[T_d]]^{S_d}$ *be a symmetric function and let*

$$f(T_d)\prod_{i<j}(t_i-t_j)=\sum_{p_i\geq 0} b(p_1,\ldots,p_d)t_1^{p_1}\cdots t_d^{p_d},\quad b(p_1,\ldots,p_d)\in\mathbb{C}.$$

Then the multiplicity series of $f(T_d)$ *is given by*

$$M(f,T_d)=\frac{1}{t_1^{d-1}t_2^{d-2}\cdots t_{d-2}^2 t_{d-1}}\sum_{p_i>p_{i+1}} b(p_1,\ldots,p_d)t_1^{p_1}\cdots t_d^{p_d},$$

where the summation is on all $p=(p_1,\ldots,p_d)$ *such that* $p_1>p_2>\cdots>p_d$.

It is very easy to check whether the formal power series

$$h(T_d) = \sum h(q_1, \dots, q_d) t_1^{q_1} \cdots t_d^{q_d}, \quad q_1 \geq \cdots \geq q_d,$$

is equal to the multiplicity series $M(f, T_d)$ of the symmetric function $f(T_d)$ because $h(T_d) = M(f, T_d)$ if and only if

$$f(T_d) \prod_{i<j} (t_i - t_j) = \sum_{\sigma \in S_d} \mathrm{sign}(\sigma) t_{\sigma(1)}^{d-1} t_{\sigma(2)}^{d-2} \cdots t_{\sigma(d-1)} h(t_{\sigma(1)}, \dots, t_{\sigma(d)}).$$

These arguments can be used to verify most of our computational results on multiplicities.

For symmetric nice rational functions in any number of variables there are algorithms to find the multiplicity series. The methods are based on ideas of Berele combined with classical results of Elliott (1903) developed to solve linear diophantine equations and inequalities in nonnegative integers, generalized later by MacMahon (1916) to Partition Analysis (or Ω-Calculus). Well forgotten, the method of Elliott and MacMahon has his second life in a series of twelve papers on MacMahon's partition analysis by Andrews, alone or jointly with Paule, Riese and Strehl, with further improvements and computer realizations by {Andrews, Paule and Riese}, Han, Xin and {Fu and Lascoux}.

To explain the method it is sufficient to consider the case of two variables only. Let $t = t_1$, $u = t_2$.

Lemma

Let $f(t,u) \in \mathbb{C}[[t,u]]^{S_2}$ be a symmetric function with multiplicity series $M(f,t,u)$ and let

$$(t-u)f(t,u) = \sum_{i,j\geq 0} b_{ij} t^i u^j, \quad b_{ij} \in \mathbb{C}.$$

Then $f(t,u)$ and $M(f,t,u)$ are related by the equations

$$f(t,u) = \frac{tM(f,t,u) - uM(f,u,t)}{t-u},$$

$$M(f,t,u) = \frac{1}{t} \sum_{i>j} b_{ij} t^i u^j.$$

Applied to our problem to compute the multiplicity series of a nice symmetric function in two variables, the approach of Elliott-MacMahon is the following. Given a symmetric function $f(t,u)\in\mathbb{C}[[t,u]]^{S_2}$, we form the series

$$g(t,u)=(t-u)f(t,u)=\sum_{i,j\geq 0}b_{ij}t^iu^j.$$

We want to compute the "half" of $g(t,u)$, for $i>j$. We introduce a new variable z and consider the Laurent series

$$g(tz,\frac{u}{z})=\sum_{i,j\geq 0}b_{ij}t^iu^jz^{i-j}=\sum_{n=-\infty}^{\infty}g_n(t,u)z^n,\quad g_n(t,u)\in\mathbb{C}[[t,u]].$$

Then the multiplicity series of $f(t,u)$ is determined from the equality

$$tM(f;t,u)=\sum_{i>j}b_{ij}t^iu^j=\sum_{n\geq 0}g_n(t,u).$$

Hence we have to present the Laurent series $g(tz, u/z)$ as a sum of two series, in z and $1/z$, respectively:

$$g(tz, \frac{u}{z}) = \sum_{n\geq 0} g_n(t,u)z^n + \sum_{n>0} g_n(t,u)\left(\frac{1}{z}\right)^n,$$

to take the first summand and to replace z with 1 there. If $f(t,u)$ is a nice symmetric function, then $g(t,u)$ and $g(tz, u/z)$ have the form

$$g(t,u) = p(t,u)\prod \frac{1}{1 - t^a u^b}, \quad p(t,u) \in \mathbb{C}[t,u],$$

$$g(tz, \frac{u}{z}) = p(tz, \frac{u}{z})\prod \frac{1}{1 - t^a u^b z^{a-b}}.$$

The expression $\prod 1/(1-t^a u^b z^{a-b})$ is a product of three factors

$$\prod_{a_0=b_0} \frac{1}{1-t^{a_0}u^{b_0}}, \quad \prod_{a_1>b_1} \frac{1}{1-t^{a_1}u^{b_1}z^{a_1-b_1}}, \quad \prod_{a_2<b_2} \frac{1}{1-t^{a_2}u^{b_2}/z^{b_2-a_2}}.$$

If $\prod 1/(1-t^a u^b z^{a-b})$ contains factors of both the second and the third type, Elliott suggests to apply the equality

$$\frac{1}{(1-Az^a)(1-B/z^b)} = \frac{1}{1-ABz^{a-b}}\left(\frac{1}{1-Az^a}+\frac{1}{1-B/z^b}-1\right)$$

to one of the expressions $1/(1-t^{a_1}u^{b_1}z^{a_1-b_1})(1-t^{a_2}u^{b_2}/z^{b_2-a_2})$ and to present $\prod 1/(1-t^a u^b z^{a-b})$ as a sum of three expressions which are simpler than the original one. Continuing in this way, one presents $\prod 1/(1-t^a u^b z^{a-b})$ as a sum of products of two types

$$\prod_{a\geq b} \frac{1}{1-t^a u^b z^{a-b}}, \quad \prod_{a_0=b_0} \frac{1}{1-t^{a_0}u^{b_0}} \prod_{a_2<b_2} \frac{1}{1-t^{a_2}u^{b_2}/z^{b_2-a_2}}.$$

With some additional easy arguments we can present $g(tz, u/z)$ as a linear combination of quotients of the form

$$B_1 z^i \prod_{b\geq 0} \frac{1}{1-Bz^b}, m \geq 0, \quad \frac{C_1}{z^j} \prod \frac{1}{1-C_2} \prod_{c>0} \frac{1}{1-C_3/z^c}, n \geq 0,$$

with coefficients B_1, B_2, C_1, C_2, C_3 which are monomials in t, u. Comparing this form of $g(tz, u/z)$ with its expansion

$$g(tz, u/z) = \sum_{n=-\infty}^{\infty} g_n(t, u) z^n$$

as a Laurent series in z we obtain that the part $\sum_{n\geq 0} g_n(t, u) z^n$ which we want to compute is the sum of $B_1 z^i \prod_{b\geq 0} 1/(1-Bz^b)$ and the fractions $C_1/z^j \prod 1/(1-C_2)$ with $j = 0$. This approach provides an easy proof of the result of Berele and gives also an algorithm to find the multiplicity series of the nice symmetric function $f(t, u)$.

There is another algorithm due to Xin which is more efficient than the original algorithm of Elliott. We use an algorithm inspired by the algorithm of Xin.

Algorithm of Xin

Let $g(t,u) \in \mathbb{C}[[t,u]]$ be a nice rational function. In $g(tz, u/z)$ we replace the factors $1/(1 - C/z^c)$, where C is a monomial in t, u, with the factor $z^c/(z^c - C)$. Then $g(tz, u/z)$ becomes a rational function of the form

$$g(tz, \frac{u}{z}) = \frac{p(z)}{z^a} \prod \frac{1}{1-A} \prod \frac{1}{1-Bz^b} \prod \frac{1}{z^c - C},$$

where $p(z)$ is a polynomial in z with coefficients which are rational functions in t, u and A, B, C are monomials in t, u.

Presenting $g(tz, u/z)$ as a sum of partial fractions with respect to z we obtain that

$$g(tz, \frac{u}{z}) = p_0(z) + \sum \frac{p_i}{z^i} + \sum \frac{r_{jk}(z)}{q_j(z)^k},$$

where $p_0(z), r_{jk}(z), q_j(z) \in \mathbb{C}(t, u)[z]$, $p_i \in \mathbb{C}(t, u)$, $q_j(z)$ are the irreducible factors over $\mathbb{C}(t, u)$ of the binomials $1 - Bz^b$ and $z^c - C$ in the expression of $g(tz, u/z)$ and $\deg_z r_{jk}(z) < \deg_z q_j(z)$. Clearly $p_0(z)$ gives a contribution to the series $\sum_{n\geq 0} g_n(t, u)z^n$ in the expansion of $g(tz, u/z)$ as a Laurent series. Similarly, $r_{jk}(z)/q_j(z)^k$ contributes to the same series for the factors $q_j(z)$ of $1 - Bz^b$.

The fraction p_i/z^i is a part of the series $\sum_{n>0} g_n(t,u)/z^n$. When $q_j(z)$ is a factor of $z^c - C$, we obtain that $q_j(z) = z^d q_j'(1/z)$, where $d = \deg_z q_j(z)$ and $q_j'(\zeta) \in \mathbb{C}(t,u)[\zeta]$ is a divisor of $1 - C\zeta^c$. Since $\deg_z r_{jk}(z) < \deg_z q_j(z)$ we derive that $r_{jk}(z)/q_j(z)^k$ contributes to $\sum_{n>0} g_n(t,u)/z^n$ and does not give any contribution to $\sum_{n\geq 0} g_n(t,u)z^n$. Hence

$$\sum_{n\geq 0} g_n(t,u)z^n = p_0(z) + \sum \frac{r_{jk}(z)}{q_j(z)^k},$$

where the sums in the right side of the equation runs on the irreducible divisors $q_j(z)$ of the factors $1 - Bz^b$ of $g(t,u)$.

Let

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \quad Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$$

be two generic 3×3 matrices. Let R be the algebra generated by X and Y. It is well known that $R \cong F_2(M_3(K))$. Let C be the pure (or commutative) trace algebra of X and Y. It is generated by all traces of products of X and Y. Let $T = CR$ be the mixed (or noncommutative) trace algebra. The algebra C is the algebra of invariants of the group $GL_3(K)$ acting by simultaneous conjugation of X and Y. Hence it can be studied with methods of classical invariant theory. The algebra T also is related with classical invariant theory. It turns out that the properties of C and T are much better than the properties of R. Since $R \subset T$, we may use T as a first approximation of R.

Obviously

$$X = \frac{1}{3}\mathrm{tr}(X) \cdot I_3 + x, \quad Y = \frac{1}{3}\mathrm{tr}(Y) \cdot I_3 + y,$$

where x and y are generic 3×3 traceless matrices and I_3 is the 3×3 identity matrix. Then

$$C = K[\mathrm{tr}(X), \mathrm{tr}(Y)] \otimes C_0, \quad T = K[\mathrm{tr}(X), \mathrm{tr}(Y)] \otimes T_0,$$

where C_0 and T_0 are, respectively, the pure and the mixed trace algebras of the generic traceless 3×3 matrices x and y.

The Hilbert series of $C = C_{32}$ and $T = T_{32}$ were calculated, respectively, by Teranishi and {Berele and Stembridge}:

$$H(C) = H(C, t, u) = \frac{1 + t^3u^3}{q_1(t, u)q_2(t, u)q_3(t, u)q_4(t, u)}$$

where

$$q_1(t, u) = (1 - t)(1 - u), \quad q_2(t, u) = (1 - t^2)(1 - tu)(1 - u^2),$$

$$q_3(t, u) = (1 - t^3)(1 - t^2u)(1 - tu^2)(1 - u^3), \quad q_4(t, u) = 1 - t^2u^2,$$

$$H(T) = \frac{1}{(1 - t)^2(1 - u)^2(1 - t^2)(1 - u^2)(1 - tu)^2(1 - t^2u)(1 - tu^2)}.$$

Since

$$H(C, t, u) = \frac{1}{(1-t)(1-u)} H(C_0, t, u),$$

$$H(T, t, u) = \frac{1}{(1-t)(1-u)} H(T_0, t, u),$$

we obtain the Hilbert series of C_0 and T_0:

$$H(C_0) = \frac{1 + t^3u^3}{q_2(t, u)q_3(t, u)q_4(t, u)},$$

$$H(T_0) = \frac{1}{(1-t)(1-u)(1-t^2)(1-u^2)(1-tu)^2(1-t^2u)(1-tu^2)}.$$

We shall illustrate the method of Xin on the Hilbert series of T_0. For convenience we introduce new variables t and $v = tu$ and consider $\mathbb{C}[[t,v]] = \mathbb{C}[[t,u]]_{\geq} \subset \mathbb{C}[[t,u]]$. There is a bijection between the formal power series in $\mathbb{C}[[t,v]]$ and the symmetric functions in $\mathbb{C}[[t,u]]^{S_2}$. For a symmetric function $f(t,u)$ we denote $M'(f,t,v) = M(f,t,u)$.

Proposition

The multiplicity series of $H(T_0) = H(T_0,t,u)$ is

$$M'(H(T_0),t,v) = \frac{(1-t^2v^2)(1-v+v^2)+tv(1-v^2)}{(1-t)(1-t^2)(1-tv)(1-v)^2(1-v^2)^2(1-v^4)}.$$

Proof

It is sufficient to check that

$$M(H(T_0), t, u) = M'(H(T_0), t, tu)$$

satisfies the condition

$$H(T_0) = \frac{tM(H(T_0), t, u) - uM(H(T_0), u, t)}{t - u}$$

This can be verified by direct computations. But we want to illustrate how to find the expression for the multiplicity series of $H(T_0)$ using the algorithm of Xin. We consider $g(t, u) = (t - u)H(T_0, t, u)$ and decompose $g(tz, u/z)$ as a sum of partial fractions with respect to z:

$$g(tz, \frac{u}{z}) = \frac{1}{(1-t^2u^2)^2}\left(\frac{a_1(t,u)}{1-tz} + \frac{a_2(t,u)}{(1-tz)^2} + \frac{a_3(t,u)}{1+tz} + \frac{a_4(t,u)}{1-t^2uz}\right.$$

$$\left. + \frac{b_1(t,u)}{z-u} + \frac{b_2(t,u)}{(z-u)^2} + \frac{b_3(t,u)}{z+u} + \frac{b_4(t,u)}{z-tu^2}\right),$$

where

$$a_1 = \frac{3t^2u^2 - 4tu - 1}{4(1-tu)^4(1+tu)}, \quad a_2 = \frac{1}{2(1-tu)^3},$$

$$a_3 = -\frac{1}{4(1+tu)(1+t^2u^2)}, \quad a_4 = \frac{t^2u^2}{(1-tu)^4(1+tu)(1+t^2u^2)},$$

$$b_1 = \frac{u(t^2u^2 + 4tu - 3)}{4(1-tu)^4(1+tu)}, \quad b_2 = -\frac{u^2}{2(1-tu)^3},$$

$$b_3 = -\frac{u}{4(1+tu)(1+t^2u^2)}, \quad b_4 = -\frac{t^3u^4}{(1-tu)^4(1+tu)(1+t^2u^2)}.$$

Hence

$$\sum_{n\geq 0} g_n z^n = \frac{1}{(1-t^2u^2)^2}\left(\frac{a_1(t,u)}{1-tz}+\frac{a_2(t,u)}{(1-tz)^2}+\frac{a_3(t,u)}{1+tz}+\frac{a_4(t,u)}{1-t^2uz}\right),$$

$g_n = g_n(t,u)$. Replacing first z by 1 we obtain $tM(H(T_0);t,u)$ and then u by v/t we complete the computing of $M'(H(T_0);t,v)$.

Corollary

The multiplicities $m_{(\lambda_1,\lambda_2)}(T_0)$ are

$$m_\lambda(T_0) = m_{1,\lambda_2} + (\lambda_1 - \lambda_2 + 1)m_{2,\lambda_2} + (-1)^{\lambda_1-\lambda_2} m_{3,\lambda_2},$$

when $2\lambda_2 < \lambda_1$,

$$m_\lambda(T_0) = m_{1,\lambda_2} + (\lambda_1 - \lambda_2 + 1)m_{2,\lambda_2} + (-1)^{\lambda_1-\lambda_2} m_{3,\lambda_2} + m_{4,2\lambda_2-\lambda_1},$$

when $\lambda_2 \leq \lambda_1 \leq 2\lambda_2$, where $m_{1q} + (p+1)m_{2q}$ is equal to

$$\frac{1}{2^5 \cdot 5!}(q+2)(q+4)(q+6)(20p(q+2) - (2q^2 - 9q - 60))$$

for q even and to

$$\frac{1}{2^5 \cdot 5!}(q+1)(q+3)(q+5)(20p(q+5) - (2q^2 - 3q - 99)),$$

for q odd;

$$m_{3q} = \begin{cases} \frac{1}{2^6}(q+4)^2, & \text{when } q \equiv 0 \pmod 4, \\ -\frac{1}{2^6}(q+3)^2, & \text{when } q \equiv 1 \pmod 4, \\ \frac{1}{2^6}(q+2)(q+6), & \text{when } q \equiv 2 \pmod 4, \\ -\frac{1}{2^6}(q+1)(q+5), & \text{when } q \equiv 3 \pmod 4, \end{cases}$$

$$m_{4q} = \begin{cases} \frac{1}{2^5 \cdot 5!}q(q+4)(2q^3+17q^2+12q-88), & q \equiv 0 \pmod 4, \\ \frac{1}{2^5 \cdot 5!}(q-1)(q+3)(q+7)(2q^2+7q-5), & q \equiv 1 \pmod 4, \\ \frac{1}{2^5 \cdot 5!}(q-2)(q+2)(q+6)(2q^2+13q+10), & q \equiv 2 \pmod 4, \\ \frac{1}{2^5 \cdot 5!}(q+1)(q+5)(2q^3+13q^2-8q-27), & q \equiv 3 \pmod 4. \end{cases}$$

The asymptotics of $m_{(\lambda_1,\lambda_2)}(T_0)$ and $m_{(\lambda_1,\lambda_2)}(C_0)$ is

$$m_\lambda(T_0) = \frac{1}{2^4 5!}\lambda_2^4(10\lambda_1 - 11\lambda_2) + \mathcal{O}((\lambda_1+\lambda_2)^4),$$

when $2\lambda_2 < \lambda_1$,

$$m_\lambda(T_0) = \frac{1}{2^4 5!}(\lambda_2^4(10\lambda_1 - 11\lambda_2) + (2\lambda_2 - \lambda_1)^5) + \mathcal{O}((\lambda_1+\lambda_2)^4),$$

when $\lambda_1 \leq 2\lambda_2$;

$$m_\lambda(C_0) = \frac{1}{9}m_\lambda(T_0) + \mathcal{O}((\lambda_1+\lambda_2)^4).$$

In the previous talk we introduced the operator

$$Y : \mathbb{C}[[T_d]]_{\geq} \to \mathbb{C}[[T_d]]_{\geq}$$

which acts on the multiplicity series $M(f, T_d)$ by the Young rule. If $g = g(T_d) \in \mathbb{C}[[T_d]]^{S_d}$ is a symmetric function, then

$$Y(M(g), T_d) = M\left(g(T_d)\prod_{i=1}^{d}\frac{1}{1-t_i}\right).$$

The multiplicity series $M(f)$ and $Y(M(f))$ are related by

$$Y(M(f), T_d) = \prod_{i=1}^{d} \frac{1}{1-t_i} \sum (-t_2)^{\varepsilon_2} \cdots (-t_d)^{\varepsilon_d} \times$$

$$\times M(f, t_1 t_2^{\varepsilon_2}, t_2^{1-\varepsilon_2} t_3^{\varepsilon_3} \ldots t_{d-1}^{1-\varepsilon_{d-1}} t_d^{\varepsilon_d}, t_d^{1-\varepsilon_d}),$$

where the summation runs on all $\varepsilon_2, \ldots, \varepsilon_d = 0, 1$.

In the previous talk we stated that

$$H(F_d(M_2(K)), T_d) = \sum_{k\geq 0} S_{(k,k)}(T_d) \prod_{i=1}^{d} \frac{1}{(1-t_i)^2}$$

$$-\left(S_{(1^3)}(T_d) + \sum_{n\geq 1} S_{(n)}(T_d)\right) \prod_{i=1}^{d} \frac{1}{1-t_i}.$$

In the language of the Young rule this gives

$$M(M_2(K), T_d) = Y^2\left(\frac{1}{1-t_1t_2} - 1\right) - Y\left(t_1t_2t_3 + \frac{1}{1-t_1}\right).$$

We also proved the theorem of Formanek for the Hilbert series of the product of two homogeneous ideals of the free algebra. A consequence of his theorem is:

Theorem

If R_1, R_2 and R are PI-algebras such that $T(R) = T(R_1)T(R_2)$, then

$$H(F_d(R)) = H(F_d(R_1)) + H(F_d(R_2))$$

$$+(t_1 + \cdots + t_d - 1)H(F_d(R_1))H(F_d(R_2)).$$

The translation in cocharacter sequences is the formula of Berele and Regev

$$\chi_n(R) = \chi_n(R_1) + \chi_n(R_2) + \chi_{(1)} \hat{\otimes} \sum_{i=0}^{n-1} \chi_i(R_1) \hat{\otimes} \chi_{n-1-i}(R_2)$$

$$- \sum_{i=0}^{n} \chi_i(R_1) \hat{\otimes} \chi_{n-i}(R_2),$$

where $\hat{\otimes}$ denotes the outer tensor product of S_n-characters.

Upper triangular matrices $U_k = U_k(K)$

(Boumova and Drensky)
$T(M_1(K)) = T(K)$ is generated by the commutator $[x_1, x_2]$ and $F_d(K) = K[X_d]$, the polynomial algebra in d variables. Hence

$$H(F_d(K)) = \prod_{i=1}^{d} \frac{1}{1 - t_i} = \sum_{n \geq 0} S_{(n)}(T_d),$$

$$M(K) = \sum_{n \geq 0} t_1^n = \frac{1}{1 - t_1}.$$

Theorem

(Petrogradsky, Boumova, Drensky) The Hilbert series $H(F_d(U_k), T_d)$ of the algebra $F_d(U_k)$ is

$$\frac{1}{t_1+\cdots+t_d-1}\left(\left(1+(t_1+\cdots+t_d-1)\prod_{i=1}^{d}\frac{1}{1-t_i}\right)^k-1\right)$$

$$=\sum_{j=1}^{k}\binom{k}{j}\left(\prod_{i=1}^{d}\frac{1}{1-t_i}\right)^j(t_1+\cdots+t_d-1)^{j-1}.$$

Corollary

The multiplicity series of U_k is

$$M(U_k; T_d) = \sum_{j=1}^{k} \sum_{q=0}^{j-1} \sum_{\lambda \vdash q} (-1)^{j-q-1} \binom{k}{j} \binom{j-1}{q} d_\lambda Y^j(T_d^\lambda),$$

where d_λ is the degree of the irreducible S_n-character χ_λ.

Theorem

(i) *If* $m_\lambda(U_k) \neq 0$, *then* $\lambda = (\lambda_1, \dots, \lambda_{2k-1})$ *is a partition in not more than* $2k-1$ *parts and* $\overline{\lambda} = (\lambda_{k+1}, \dots, \lambda_{2k-1})$ *is a partition of* $i \leq k-1$.
(ii) *If* $\overline{\lambda}$ *is a partition of* $k-1$, *then*

$$m_\lambda(U_k) = d_{\overline{\lambda}} \dim(W_k(\lambda_1, \dots, \lambda_k)),$$

where $d_{\overline{\lambda}}$ *is the degree of the* S_{k-1}*-character* $\chi_{\overline{\lambda}}$ *and*

$$\dim(W_k(\lambda_1, \dots, \lambda_k)) = S_\lambda(\underbrace{1, \dots, 1}_{k \text{ times}}) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Theorem

The multiplicities of the cocharacter sequence of the algebra U_k of the $k \times k$ upper triangular matrices for $k = 1, 2$ are

$$m_\lambda(U_1) = \begin{cases} 1, & \lambda = (\lambda_1) \\ 0, & \lambda_2 > 0; \end{cases}$$

$$m_\lambda(U_2) - m_\lambda(U_1) = \begin{cases} \lambda_1 - \lambda_2 + 1, & \lambda = (\lambda_1, \lambda_2), \lambda_2 > 0, \\ \lambda_1 - \lambda_2 + 1, & \lambda = (\lambda_1, \lambda_2, 1), \\ 0, & \text{for all other } \lambda. \end{cases}$$

The results for U_3 are the following.

Theorem

The difference of the multiplicities $m_\lambda(U_3) - m_\lambda(U_2)$ *of* U_3 *and* U_2 *is*

$$\begin{cases} n_\lambda, & \lambda = (\lambda_1, \lambda_2, \lambda_3, 2), \\ n_\lambda, & \lambda = (\lambda_1, \lambda_2, \lambda_3, 1, 1), \\ 4n_\lambda - c_\lambda, & \lambda = (\lambda_1, \lambda_2, \lambda_3, 1), \\ 4n_\lambda - c_\lambda, & \lambda = (\lambda_1, \lambda_2, \lambda_3), \lambda_3 > 0, \\ \frac{1}{2}\lambda_1(\lambda_1 - \lambda_2 + 1)(\lambda_2 - 1), & \lambda_2 \geq 2, \\ 0, & \text{for all other } \lambda. \end{cases}$$

Here

$$n_\lambda = \dim(W_3(\lambda_1, \lambda_2, \lambda_3)) = \frac{1}{2}(\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_1 - \lambda_3 + 2)$$

and the correction c_λ is

$$c_\lambda = \begin{cases} \frac{1}{2}(\lambda_1 + 2)(\lambda_1 - \lambda_2 + 1)(\lambda_2 + 1), & \lambda = (\lambda_1, \lambda_2, 1, 1), \\ \frac{1}{2}(\lambda_1 + 3)(\lambda_1 - \lambda_2 + 1)(\lambda_2 + 2), & \lambda = (\lambda_1, \lambda_2, 1), \\ 0, & \text{for all other } \lambda. \end{cases}$$

The colength sequence of a PI-algebra R is defined as the sequence of the number of irreducible characters, counting the multiplicities, in the cocharacter sequence of R:

$$cl_n(R) = \sum_{\lambda \vdash n} m_\lambda(R), \quad n = 0, 1, 2, \ldots.$$

If the algebra R is finite dimensional then the generating function of the colength sequence, the colength series of R, can be obtained immediately from the multiplicity series $M(R; T_d)$ for a sufficiently large d:

$$cl(R; t) = \sum_{n \geq 0} cl_n(R) t^n = M(R; \underbrace{t, \ldots, t}_{d \text{ times}}).$$

The above results give:

Corollary

$$cl(U_1;t)=\frac{1}{1-t};$$

$$cl(U_2;t)-cl(U_1;t)=\frac{t^2}{(1-t)^3};$$

$$cl(U_3;t)-cl(U_2;t)=\frac{t^4(3+6t+4t^2-2t^3-t^4)}{(1-t)^3(1-t^2)^3};$$

$$cl(U_4;t)-cl(U_3;t)=\frac{t^6p(t)}{(1-t)^4(1-t^2)^6},$$

$$p(t)=11+45t+63t^2-t^3-42t^4-24t^5+16t^6+12t^7-3t^8-t^9.$$

Upper triangular 2×2 matrices with Grassmann entries

(Lucio Centrone)

$$R = \begin{pmatrix} E & E \\ 0 & E \end{pmatrix}.$$

The cocharacter sequence of the identities of E is

$$\chi_n(E) = \sum_{k=0}^{\lceil n/2 \rceil} \chi_{(1^{2k})} \hat{\otimes} \chi_{(n-2k)}, \quad n = 0, 1, 2, \ldots.$$

The outer tensor multiplication by $\chi_{(1^{2k})}$ is obtained by the "transpose" of the Young rule. Hence in the formula of Berele and Regev one applies only the Young rule, its transpose and the branching theorem. The concrete results are in the preprint of Centrone in arXiv.

Upper triangular matrices of any size with Grassmann entries

(Centrone and Drensky)
Since the cocharacters are in a hook (and not in a strip, as for finite dimensional algebras) to calculate the generating function of the multiplicities we need two sets of variables (something like for super algebras). The first set of variables counts the part of the Young diagrams which is in the arm of the hook and the other set of variables counts the part in the leg of the hook. The statement of the results is in the language of the Young operator and its transpose. Explicit formulas are found for small size of matrices.

Block triangular matrices with 1×1 and 2×2 blocks

(Drensky and Boyan Kostadinov)
Let

$$R_{p,q} = \begin{pmatrix} M_{d_1}(K) & & & * \\ 0 & M_{d_2}(K) & & \\ \vdots & & \ddots & \\ 0 & \dots & 0 & M_{d_{p+q}}(K) \end{pmatrix}$$

be the algebra of upper block triangular matrices, where p of the matrix algebras are of size 1×1 and the other q are of size 2×2. The T-ideal of $R_{p,q}$ is

$$T(R_{p,q}) = T(M_{d_1}(K)) \cdots T(M_{d_{p+q}}(K)).$$

Since the Hilbert series of $F_d(M_1(K))$ and $F_d(M_2(K))$ and hence of $F_d(R_{p,q}))$ involve only operations with

$$\sum_{k\geq 0} S_{(k)}(T_d), \quad S_{(1)}(T_d), \quad S_{(1,1,1)}(T_d),$$

(application of the the Young rule only) and

$$f(T_d) = \sum_{k\geq 0} S_{(k,k)}(T_d),$$

the only principal difficulties in the calculation of the multiplicities of $R_{P,q}$ are related with the computing of the multiplicity series of $f^m(T_d)$, $m \leq q$.

The following result for $M(f^2)$ were obtained with the Littlewood-Richardson rule as well as with the algorithm of Xin:

$$M(f) = \frac{1}{1 - t_1 t_2},$$

$$M(f^2) = \frac{1}{(1 - t_1^2 t_2 t_3)(1 - t_1 t_2)^2 (1 - t_1 t_2 t_3 t_4)}.$$

We have also explicit results for the multiplicities $m(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ in $M(f^3)$ (equivalently, for $M'(f^3)(u_1, u_2, u_3, u_4)$):

$$\frac{(1 - u_1 u_2 u_3)(1 - u_1^2 u_3^2 u_4)}{(1 + u_3)(1 - u_1^2 u_4)(1 - u_1 u_3)^3 (1 - u_3)(1 - u_2)^3 (1 - u_4)^3},$$

where $u_i = t_1 \cdots t_i$, $i = 1, 2, 3, 4$.

For $\lambda = (\lambda_1, \ldots, \lambda_d)$ we denote

$$n_i = \lambda_i - \lambda_{i+1}, \quad i = 1, \ldots, d-1, \quad n_d = \lambda_d.$$

We give the asymptotics of $m_\lambda(R_{p,q})$ for small p and q.

If $p=1$ and $q=1$:

$\lambda=(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$:

$$m_\lambda=\frac{3n_1n_2n_3n_4}{4}\left(\frac{n_1n_3n_4+n_1n_2n_4+n_2n_3n_4+n_1n_2n_3}{2}+\right.$$

$$\left.+\frac{n_2^2n_4+n_2n_3^2+n_1n_3^2+n_2^2n_3}{3}\right)+\mathcal{O}\left((n_1+n_2+n_3+n_4)^6\right),$$

$\lambda=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)$, where $\lambda_5>0$:

$$m_\lambda=n_1n_2n_3n_4\left(\frac{n_1n_3n_4+n_1n_2n_4+n_2n_3n_4+n_1n_2n_3}{2}+\right.$$

$$\left.+\frac{n_2^2n_4+n_2n_3^2+n_1n_3^2+n_2^2n_3}{3}\right)+\mathcal{O}\left((n_1+n_2+n_3+n_4)^6\right),$$

$\lambda=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,1)$:

$$m_\lambda=\frac{n_1n_2n_3n_4}{4}\left(\frac{n_1n_3n_4+n_1n_2n_4+n_2n_3n_4+n_1n_2n_3}{2}+\right.$$

$$\left.+\frac{n_2^2n_4+n_2n_3^2+n_1n_3^2+n_2^2n_3}{3}\right)+\mathcal{O}\left((n_1+n_2+n_3+n_4)^6\right).$$

For $p = 0$, $q = 2$ and $\lambda = (\lambda_1, \lambda_2)$:

$$m_\lambda = \frac{1}{6!} n_1 n_2^4 (5n_1^2 + 6n_1 n_2 + 2n_2^2) + \mathcal{O}\left((n_1 + n_2)^6\right).$$

For $p = 0$, $q = 2$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $n_3 \geqq n_1$:

$$m_\lambda = 4\frac{n_1^2 n_2^5 n_3^5}{2!5!5!} + 16\frac{n_1^2 n_2^2 n_3^8}{2!2!8!} + 6\frac{n_1^2 n_2 n_3^9}{2!1!9!} + 18\frac{n_1^2 n_2^3 n_3^7}{2!3!7!} + 12\frac{n_1^2 n_2^4 n_3^6}{2!4!6!} +$$

$$+2\frac{n_2(n_3 - n_1)^{11}}{1!11!} - 2\frac{n_2 n_3^{11}}{1!11!} + 10\frac{n_1^3 n_2 n_3^8}{3!1!8!} + 10\frac{n_1^3 n_2^2 n_3^7}{3!2!7!} + 6\frac{n_1^3 n_2^3 n_3^6}{3!3!6!} + 2\frac{n_1^3 n_2^4 n_3^5}{3!4!5!} +$$

$$+8\frac{n_1 n_2^2 n_3^9}{1!2!9!} + 18\frac{n_1 n_2^4 n_3^7}{1!4!7!} + 4\frac{n_1 n_2^6 n_3^5}{1!6!5!} + 16\frac{n_1 n_2^3 n_3^8}{1!3!8!} + 2\frac{n_1 n_2 n_3^{10}}{1!1!10!} + 12\frac{n_1 n_2^5 n_3^6}{1!5!6!} +$$

$$+\mathcal{O}\left((n_1 + n_2 + n_3)^{11}\right).$$

For $p = 0$, $q = 3$ and $\lambda = (\lambda_1, \lambda_2)$:

$$m_\lambda = \frac{1}{11!} n_1 n_2^7 (66 n_1^4 + 77 n_1 n_2^3 + 165 n_1^2 n_2^2 + 165 n_1^3 n_2 + 14 n_2^4) + \mathcal{O}\left((n_1 + n_2)^{11}\right.$$

For $p = 0$, $q = 4$ and $\lambda = (\lambda_1, \lambda_2)$:

$$m_\lambda = \frac{1}{10!} n_1 n_2^6 (n_1 + n_2)^6 + \mathcal{O}\left((n_1 + n_2)^{12}\right),$$

Let

$$c_0(R), c_1(R), c_2(R), \ldots$$

be the codimension sequence of the PI-algebra R. We consider the generating function

$$c(R,t) = \sum_{n\geq 0} c_n(R)t^n$$

and the exponential generating function

$$\tilde{c}(R,t) = \sum_{n\geq 0} c_n(R)\frac{t^n}{n!}.$$

It is well known that $c(R,t)$ is a rational function if and only if the sequence $c_n(R)$, $n = 0, 1, 2, \ldots$, satisfies a linear homogeneous recurrence relation. There exists a positive integer k and complex numbers $p_1, \ldots, p_k$ such that

$$c_{n+k}(R) = p_1 c_{n+k-1} + p_2 c_{n+k-2}(R) + \cdots + p_k c_n(R), \quad n = 0, 1, 2, \ldots.$$

This is equivalent to the fact that $\tilde{c}(R,t)$ satisfies a linear homogeneous differential equation with constant coefficients.

Lemma

(Drensky, Petrogradsky, implicitly Berele and Regev) If $T(R) = T(R_1)T(R_2)$, then

$$\tilde{c}(R,t) = \tilde{c}(R_1,t) + \tilde{c}(R_2,t) + (t-1)\tilde{c}(R_1,t)\tilde{c}(R_2,t).$$

Corollary

(Boumova, Drensky) If $T(R) = T(R_1)T(R_2)$ and $c(R_1,t)$ and $c(R_2,t)$ are rational functions, then $c(R,t)$ is also rational.

Proof. If $c(R_1,t)$ and $c(R_2,t)$ are rational, then $\tilde{c}(R_1,t)$ and $\tilde{c}(R_2,t)$ (as solutions of linear homogeneous differential equations with constant coefficients) are linear combinations of functions of the form $t^m e^{at}$ and the same holds for $\tilde{c}(R_1,t)\tilde{c}(R_2,t)$ and hence for $\tilde{c}(R,t)$. The latter implies that $c(R,t)$ is rational.

More nontrivial result:

Theorem

(Boumova, Drensky) If $T(R) = T(R_1)T(R_2)$, $c(R_1, t)$ is rational and $c(R_2, t)$ is algebraic over $\mathbb{C}(t)$, then $c(R, t)$ is also algebraic.

Remark

An old theorem of H.A. Jungen (1931) states: Let $F(x) = \sum_{n\geq 0} f(n)x^n$ and $f(n) \sim Cn^{-g}a^n$ as $n \to \infty$, where C and a are complex and g is real. If $F(x)$ is algebraic, then g is rational; if, in addition $g > 0$, then g is not an integer. Regev computed the exact asymptotic behaviour of the codimension sequence $c_n(M_p(K))$, $n = 0, 1, 2, \ldots$. Using the above theorem, Beckner and Regev showed that for $p \geq 3$ odd, the asymptotics of $c_n(M_p(K))$ implies that $c(M_p(K), t)$ is not algebraic. When $p \geq 4$ is even, this is an open problem, and Regev conjectured that again $c(M_p(K), t)$ is not algebraic.

Examples of algebraic codimension series

$$c(M_2(K),t)=\frac{1}{2t^2}(1-2t-\sqrt{1-4t})-\frac{t^3}{(1-t)^4}+\frac{1}{1-t}-\frac{1}{1-2t},$$

$$c_n(M_2(K))=\frac{1}{n+2}\binom{2n+2}{n+1}-\binom{n}{3}+1-2^n,\quad c_n(M_2(K))\sim\frac{4^{n+1}}{n\sqrt{\pi n}};$$

$$c(E\otimes E,t)=\frac{1}{2}+\frac{1}{2\sqrt{1-4t}}+\frac{t}{(1-t)^2}+\frac{1}{1-t}-\frac{1}{1-2t},$$

$$c_n(E\otimes E)=\frac{1}{2}\binom{2n}{n}+n+1-2^n,\quad n>0,\quad c_n(E\otimes E)\sim\frac{4^n}{2\sqrt{\pi n}}.$$

Weak polynomial identities in two variables for $M_3(K)$

(Centrone, Drensky, Roberto La Scala)
Let $T(M_p, sl_p) = T(M_p(K), sl_p(K))$ be the ideal of weak polynomial identities of the pair $(M_p(K), sl_p(K))$ and let

$$(F(M_p, sl_p), L(sl_p)) = (K\langle X\rangle/T(M_p, sl_p), L/T(sl_p))$$

where L is the free Lie algebra canonically embedded into $K\langle X\rangle$. Following the scheme of Razmyslov for 2×2 matrices, the understanding of $F(M_p, sl_p)$ is the first step in the understanding of the Lie polynomial identities of sl_p. (It is necessary to describe the Lie elements in $F(M_p, sl_p)$.) If we describe the proper polynomials in $F(M_p, sl_p)$, this would give us information for the ordinary polynomial identities of M_p.

Let R_0, C_0, T_0 be, respectively, the algebra generated by two generic traceless 3×3 matrices x and y, and the related pure and mixed trace algebras. The structure of T_0 is described by Benanti and Drensky. Let S_0 be the algebra generated by

$$\mathrm{tr}(x^2), \mathrm{tr}(xy), \mathrm{tr}(y^2), \mathrm{tr}(x^3), \mathrm{tr}(x^2y), \mathrm{tr}(xy^2), \mathrm{tr}(y^3), \mathrm{tr}(x^2y^2 - xyxy).$$

It is known that S_0 is a polynomial algebra generated by these 8 traces and as an S_0-module

$$C_0 = S_0 \oplus S_0 \mathrm{tr}(x^2y^2xy - y^2x^2yx).$$

Theorem

(Benanti and Drensky) The algebra T_0 is a free S_0-module of rank 18. It has an explicitly given minimal generating system which spans a $GL_2(K)$-module

$$G = W(0,0) \oplus W(1,0) \oplus W(2,0) \oplus W(1,1)$$

$$\oplus 2W(2,1) \oplus W(3,1) \oplus W(2,2) \oplus W(3,2) \oplus W(3,3),$$

where $W(\lambda_1, \lambda_2)$ is the GL_2-module corresponding to the partition (λ_1, λ_2).

Theorem

Let $R_0' = R_0[R_0, R_0]R_0$ be the commutator ideal of R_0. Then R_0' is an S_0-module. It is a direct sum of the free S_0-module of rank 12 generated by the GL_2-submodule

$$W(1,1) \oplus 2W(2,1) \oplus W(3,1) \oplus W(2,2) \oplus W(3,2) \oplus W(3,3)$$

and of

$$Q = R_0' \cap (S_0 \oplus S_0 x \oplus S_0 y \oplus S_0 x^2 \oplus S_0(xy + yx) \oplus S_0 y^2).$$

Theorem

The S_0-module Q has a minimal generating system which spans the GL_2-module

$$W(3,1) \oplus W(2,2) \oplus W(4,1) \oplus 2W(3,2) \oplus 2W(4,2).$$

As an S_0-module Q is not free and has 18 defining relations of degree 8 and 9. The Hilbert series of the GL_2-modules T_0/R_0' and T_0/R_0 are

$$H(T_0/R_0', t, u) = \frac{1}{(1-t)(1-u)(1-t^2)(1-tu)(1-u^2)},$$

$$H(T_0/R_0, t, u) = \frac{t^3u^3}{(1-t)(1-u)(1-t^2)(1-tu)(1-u^2)}$$

$$+\frac{t^2+tu+u^2}{(1-t)(1-u)(1-t^2)(1-u^2)}.$$

Conclusion

The multiplicities $m_\lambda(R_0)$ are almost the same as the multiplicities of $m_\lambda(T_0)$.

Remark

Similar results are obtained for the center of R_0' as an S_0-module. Again, the multiplicities $m_\lambda(Z(R_0))$ of the centre $Z(R_0)$ of R_0 are almost the same as the multiplicities of $m_\lambda(C_0)$.