The Kostrikin radical in characteristic zero

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(joint work with M. Gómez Lozano)
What do we know in char 0?

Condition $\hat{\mathcal{H}}$

$\forall$ quotient of $L$, $\forall$ (no limit) ordinal $\beta$,

$M$ submodule of $K_\beta(L)/K_{\beta-1}(L)$

$M$ invariant under inner automorphisms \{ $\implies$ $M$ ideal \}

Theorem

$L$ nondegenerate, $\hat{\mathcal{H}}$, every ideal with Jordan elements.

Then $K(L) = \bigcap$ (str. prime ideals of $L$) $= 0$.

Main example

$L$ over a field of characteristic zero $\implies \hat{\mathcal{H}}$
Can we do better?

Suggestion by Efim Zelmanov:

USE GENERALIZED M-SEQUENCES!!

(e-mails and manuscript notes exchanged with E. Zelmanov)

Theorem

$L$ over a field of characteristic zero

- $K(L) = \{ a \in L \mid \text{finite generalized m-sequence} \}$
- $K(L) = \bigcap (\text{str. prime ideals of } L)$
- $L$ nondegenerate $\Rightarrow L$ subdirect product of str. prime algebras.
The sets $B_n(L)$: $K_1(L) = \bigcup_n B_n(L)$

$$B_n(L) = \left\{ \sum_{i=1}^n [[[a_i, b_{i1}], \ldots, b_{ik_i}]] \mid 0 \leq k_i \leq n, \ b_{ij} \in L \right\}$$

$$(\text{ad}_{a_i}^2 = 0, \ i = 1, \ldots, n)$$

The functions $f(n, r)$:

Lemma

$$\forall n, r \in \mathbb{N} \ \exists f(n, r) \in \mathbb{N} \text{ such that } \forall L \text{ of char } 0 \text{ and } \forall a \in B_n(L)$$

$$\text{ad}_{[[a,b_1],\ldots,b_k]}^{f(n,r)} b_0 = 0 \text{ for every } b_0, \ldots, b_k \in L, \ 0 \leq k \leq r.$$ 

Important:

- $f(n, r)$ exists with independence of the Lie algebra
- $n$ for the set $B_n(L)$, $r$ for how many arbitrary elements $b_i \in L$
**Construction of the \( f(n, r) \):**

We want \( \text{ad}^{f(n,r)}_{[[a,b_1],\ldots,b_k]} b_0 = 0 \)

Remember that \( a \in B_n(L) \) has the form \( \sum_{i=1}^{n} [[[a_i, b_{i_1}], \ldots, b_{i_{k_i}}]] \)

New variables:

\[
X := \{ x_0 \} \cup \{ x_i \mid i \in \mathbb{N} \} \cup \{ x_{ij} \mid i, j \in \mathbb{N} \} \cup \{ y_i \mid i \in \mathbb{N} \}
\]

- \( x_0 \) for where we evaluate, “\( b_0 \)”
- \( x_i \) for the absolute zero divisors “\( a_i \)” in the construction of “\( a \)”
- \( x_{ij} \) for the “arbitrary” elements “\( b_{ij} \)” in the construction of “\( a \)”
- \( y_i \) for the “arbitrary” elements “\( b_i \)”

\( L[X] \) free Lie algebra,

\[
\bar{L}[X] = L[X]/\text{Id}_{L[X]}(\text{ad}_{x_i}^2 L[X] \mid i \in \mathbb{N})
\]
Construction of the $f(n, r)$:
We want $\text{ad}^{f(n, r)}_{[[a, b_1], \ldots, b_k]} b_0 = 0, \quad a = \sum_{i=1}^{n} [[[a_i, b_{i1}], \ldots, b_{ik_i}]]$
\[\bar{L}[X]\] “free” with absolute zero divisors “$x_i$”

To imitate elements of the form $[[a, b_1], \ldots, b_k]$ we build the sets:

$$A_{n,r} := \left\{ \sum_{i=1}^{n} [[[\bar{x}_i, \bar{x}_{i1}], \ldots, \bar{x}_{ik_i}], \bar{y}_1], \ldots, \bar{y}_k] \mid 0 \leq k_i \leq n, 0 \leq k \leq r \right\}. $$

**Important:** $A_{n,r} \subset K_1(\bar{L}[X])$ and **finite** $\iff$ also $A_{n,r} \cup [A_{n,r}, x_0]$
$\implies$ $A_{n,r} \cup [A_{n,r}, x_0]$ generates a **nilpotent** subalgebra $D_{n,r}$
$\downarrow$
$\exists f(n, r) \in \mathbb{N}$ such that $D_{n,r}^{f(n, r)} = 0$ (rmk: $f(n, r) \geq 3$)
Generalized m-sequence:

**Definition**

\( \{c_i\}_{i \in \mathbb{N}} \) such that \( c_1 \in L \) and each \( c_{i+1} \) has form

\[
\text{ad}_{c_i}^{q_i} x_0, \quad \text{ad}_{[c_i, x_1]}^{q_i} x_0, \quad \text{or} \quad \text{ad}_{[[c_i, x_1], x_2]}^{q_i} x_0
\]

for some \( x_0, x_1, x_2 \in L \) and \( q_i = f(i, 3i + 2) \) ← (technical)

**Remark:**

\[
\begin{align*}
\text{ad}_{c_i}^{q_i} x_0 & \in [c_i, [c_i, [c_i, L]]] \subset [[[c_i, L], L], L] \\
\text{ad}_{[c_i, x_1]}^{q_i} x_0 & \in [[[c_i, x_1], [c_i, x_1], L]] \subset [[[c_i, L], L], L] \\
\text{ad}_{[[c_i, x_1], x_2]}^{q_i} x_0 & \in [[[c_i, x_1], x_2], L] \subset [[[c_i, L], L], L]
\end{align*}
\]
Why generalized m-sequences?

Proposition 1

\[ \{c_i\}_{i \in \mathbb{N}} \text{ with some } c_i \in K(L) \Rightarrow \text{finite length} \]

Proof: (transfinite induction)

if \( c_i \in K_1(L) \), \( c_i \) in some \( B_n(L) \) (assume \( n \geq i \))

\[ c_{i+1} \in [[[c_i, L], L], L], \quad c_{i+2} \in [[[c_{i+1}, L], L], L] \subset [[[c_i, L], \ldots, L]], \]

so \( c_n \in [[[c_i, L], \ldots, L]] \) hence \( c_{n+1} = 0 \)

\[ (\text{since } q_n = f(n, 3n + 2)) \]
Why three types of elements in a generalized m-sequence?

Proposition 2
If \( a \in L \) and \( \exists q \in \mathbb{N} \) with
\[
\text{ad}_a^q x_0 = \text{ad}_{[a,x_1]}^q x_0 = \text{ad}^{q}_{[[a,x_1],x_2]} x_0 = 0, \quad \text{for all } x_0, x_1, x_2 \in L
\]
then \( a \in K(L) \).

Proof: (work in \( L/K(L) \), assume \( L \) nondegenerate and \( a \neq 0 \))
- either \( b = a \) or \( b \in [a, L] \) has index of ad-nilpotency 3
- consider \( L_b = (L/\ker(b), \cdot) \), \( x \cdot y := \frac{1}{2} [[x, b], y] \)
- \( L_b \) is nilpotent of index \( \leq q + 1 \) \( (\tilde{x}^{q+1,b} = \frac{1}{2^{q+1}} \text{ad}^q_{[x,b]} x) \)
- \( L_b \) radical in the sense of McCrimmon + \( L_b \) nondegenerate
- \( L_b = 0 \implies L = \ker(b) \), \([b, [b, L]] = 0 \) CONTRADICTION

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Proposition 3

\[ a \notin K(L) \implies \exists \text{ infinite } \{c_i\}_{i \in \mathbb{N}} \text{ with } c_0 = a \]

Proof: (work in \( L/K(L) \), \( a \neq 0 \))

If \( c_i \neq 0 \) then \( c_{i+1} \neq 0 \):

otherwise \( \text{ad}^{q_i}_{c_i}x_0 = \text{ad}^{q_i}_{[c_i,x_1]}x_0 = \text{ad}^{q_i}_{[[c_i,x_1],x_2]}x_0 = 0 \implies c_i = 0 \)

Theorem 1 (Proposition 1 + Proposition 3)

\[ K(L) = \{ a \in L \mid \text{every } \{c_i\}_{i \in \mathbb{N}} \text{ with } c_0 = a \text{ is finite} \} \]
Remember $P$ str. prime ideal if $L/P$ is prime and nondegenerate.

**Proposition 4**

\[
\{c_i\}_{i \in \mathbb{N}} \text{ infinite and } P \text{ maximal with respect to } P \cap \{c_i\}_{i \in \mathbb{N}} = \emptyset
\]

\[\downarrow\]

$P$ is str. prime ideal

(notice that each $c_{i+1} \in [\text{Id}_L(c_i), \text{Id}_L(c_i)]$)

**Theorem 2**

\[K(L) = \bigcap (\text{str. prime ideals of } L)\]

**Corollary**

$L$ nondegenerate $\implies$ subdirect product of str. prime algebras.
References:

- E. García, M. Gómez Lozano “A characterization of the Kostrikin radical of a Lie algebra” (preprint) http://homepage.uibk.ac.at/~c70202/jordan/ [287].