The Kostrikin radical in characteristic zero

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Newfoundland, July 18-23, 2010

(joint work with M. Gómez Lozano)

What do we know in char 0?

Condition $\hat{\mathcal{H}}$

$\forall$ quotient of L, $\forall$ (no limit) ordinal β,

$$\left.\begin{array}{l} M \text{ \underline{submodule} of } K_\beta(L)/K_{\beta-1}(L) \\ M \text{ invariant under inner automorphisms} \end{array}\right\} \Longrightarrow M \text{ \underline{ideal}}$$

Theorem

L nondegenerate, $\hat{\mathcal{H}}$, every ideal with Jordan elements.
Then $K(L) = \bigcap(\text{str. prime ideals of } L) = 0$.

Main example

L over a field of characteristic zero $\Longrightarrow \hat{\mathcal{H}}$

Can we do better?

Suggestion by Efim Zelmanov:

USE GENERALIZED M-SEQUENCES!!

(e-mails and manuscript notes exchanged with E. Zelmanov)

$\Downarrow\Downarrow\Downarrow$

Theorem

L over a field of characteristic zero

- $K(L) = \{a \in L \mid \text{finite generalized m-sequence}\}$
- $K(L) = \bigcap(\text{str. prime ideals of } L)$
- L nondegenerate $\Rightarrow$ L subdirect product of str. prime algebras.

The sets $B_n(L)$: $\quad K_1(L) = \bigcup_n B_n(L)$

$$B_n(L) = \{\sum_{i=1}^{n} [[[a_i, b_{i_1}], \dots, b_{i_{k_i}}]] \mid 0 \leq k_i \leq n,\ b_{i_j} \in L\}$$

$$(\mathrm{ad}_{a_i}^2 = 0,\ i = 1, \dots, n)$$

The functions $f(n,r)$:

Lemma

$\forall n, r \in \mathbb{N}\ \ \exists f(n,r) \in \mathbb{N}$ such that $\forall L$ of char 0 and $\forall a \in B_n(L)$

$$\mathrm{ad}_{[[a,b_1],\dots,b_k]}^{f(n,r)} b_0 = 0 \text{ for every } b_0, \dots, b_k \in L,\ 0 \leq k \leq r.$$

Important:

- $f(n,r)$ exists with independence of the Lie algebra
- n for the set $B_n(L)$, r for how many arbitrary elements $b_i \in L$

Construction of the $f(n, r)$:

We want $\mathrm{ad}^{f(n,r)}_{[[a,b_1],\ldots,b_k]} b_0 = 0$

Remember that $a \in B_n(L)$ has the form $\sum_{i=1}^{n} [[[a_i, b_{i_1}], \ldots, b_{i_{k_i}}]]$

New variables:

$$X := \{x_0\} \cup \{x_i \mid i \in \mathbb{N}\} \cup \{x_{ij} \mid i, j \in \mathbb{N}\} \cup \{y_i \mid i \in \mathbb{N}\}$$

- x_0 for where we evaluate, "b_0"
- x_i for the absolute zero divisors "a_i" in the construction of "a"
- x_{ij} for the "arbitrary" elements "b_{ij}" in the construction of "a"
- y_i for the "arbitrary" elements "b_i"

$\mathcal{L}[X]$ free Lie algebra,

$$\bar{\mathcal{L}}[X] = \mathcal{L}[X]/\mathrm{Id}_{\mathcal{L}[X]}(\mathrm{ad}^2_{x_i}\mathcal{L}[X] \mid i \in \mathbb{N})$$

Construction of the $f(n,r)$:

We want $\mathrm{ad}^{f(n,r)}_{[[a,b_1],\dots,b_k]} b_0 = 0$, $a = \sum_{i=1}^{n} [[[a_i, b_{i_1}], \dots, b_{i_{k_i}}]]$

$\bar{\mathcal{L}}[X]$ "free" with absolute zero divisors "x_i"

To imitate elements of the form $[[a, b_1], \dots, b_k]$ we build the sets:

$$A_{n,r} := \{\sum_{i=1}^{n} [[[[\bar{x}_i, \bar{x}_{i1}], \dots, \bar{x}_{ik_i}], \bar{y}_1], \dots, \bar{y}_k] \mid 0 \le k_i \le n,\ 0 \le k \le r\}.$$

Important: $A_{n,r} \subset K_1(\bar{\mathcal{L}}[X])$ and **finite** $\implies$ also $A_{n,r} \cup [A_{n,r}, x_0]$
$\implies$ $A_{n,r} \cup [A_{n,r}, x_0]$ generates a nilpotent subalgebra $D_{n,r}$

$\Downarrow$

$\exists f(n,r) \in \mathbb{N}$ such that $D_{n,r}^{f(n,r)} = 0$ (rmk: $f(n,r) \ge 3$)

Generalized m-sequence:

Definition

$\{c_i\}_{i\in\mathbb{N}}$ such that $c_1 \in L$ and each c_{i+1} has form

$$\mathrm{ad}_{c_i}^{q_i} x_0, \ \ \mathrm{ad}_{[c_i,x_1]}^{q_i} x_0, \ \text{ or } \mathrm{ad}_{[[c_i,x_1],x_2]}^{q_i} x_0$$

for some $x_0, x_1, x_2 \in L$ and $q_i = f(i, 3i+2)$ $\quad\leftarrow$ (technical)

Remark:

$$\mathrm{ad}_{c_i}^{q_i} x_0 \in [c_i, [c_i, [c_i, L]] \subset [[[c_i, L], L], L]$$
$$\mathrm{ad}_{[c_i,x_1]}^{q_i} x_0 \in [[c_i, x_1], [[c_i, x_1], L]] \subset [[[c_i, L], L], L]$$
$$\mathrm{ad}_{[[c_i,x_1],x_2]}^{q_i} x_0 \in [[[c_i, x_1], x_2], L] \subset [[[c_i, L], L], L]$$

Why generalized m-sequences?

Proposition 1

$\{c_i\}_{i\in\mathbb{N}}$ with some $c_i \in K(L) \Rightarrow$ finite length

Proof: (transfinite induction)

if $c_i \in K_1(L)$, c_i in some $B_n(L)$ (assume $n \geq i$)

$$c_{i+1} \in [[[c_i, \underbrace{L], L], L}_{3}], \quad c_{i+2} \in [[[c_{i+1}, \underbrace{L], L], L}_{3}] \subset [[[c_i, \underbrace{L], \ldots, L}_{3\cdot 2}]],$$

$$\text{so } c_n \in [[[c_i, \underbrace{L], \ldots, L}_{3(n-i)}]] \text{ hence } c_{n+1} = 0$$

$$(\text{since } q_n = f(n, 3n+2))$$

Why three types of elements in a generalized m-sequence?

Proposition 2

If $a \in L$ and $\exists q \in \mathbb{N}$ with

$$\mathrm{ad}_a^q x_0 = \mathrm{ad}_{[a,x_1]}^q x_0 = \mathrm{ad}_{[[a,x_1],x_2]}^q x_0 = 0, \qquad \text{for all } x_0, x_1, x_2 \in L$$

then $a \in K(L)$.

Proof: (work in $L/K(L)$, assume L nondegenerate and $a \neq 0$)

- either $b = a$ or $b \in [a, L]$ has index of ad-nilpotency 3
- consider $L_b = (L/\ker(b), \bullet)$, $x \bullet y := \frac{1}{2}[[x,b],y]$
- L_b is nilpotent of index $\leq q+1$ $\qquad (\bar{x}^{(q+1,b)} = \frac{1}{2^{n-1}}\overline{\mathrm{ad}_{[x,b]}^q x})$
- L_b radical in the sense of McCrimmon + L_b nondegenerate
- $L_b = 0 \implies L = \ker(b)$, $[b,[b,L]] = 0$ CONTRADICTION

Proposition 3

$a \notin K(L) \implies \exists$ <u>infinite</u> $\{c_i\}_{i\in\mathbb{N}}$ with $c_0 = a$

<u>Proof</u>: (work in $L/K(L)$, $a \neq 0$)

If $c_i \neq 0$ then $c_{i+1} \neq 0$:

otherwise $\mathrm{ad}_{c_i}^{q_i} x_0 = \mathrm{ad}_{[c_i,x_1]}^{q_i} x_0 = \mathrm{ad}_{[[c_i,x_1],x_2]}^{q_i} x_0 = 0 \implies c_i = 0$

Theorem 1 (Proposition 1 + Proposition 3)

$$K(L) = \{a \in L \mid \text{every } \{c_i\}_{i\in\mathbb{N}} \text{ with } c_0 = a \text{ is } \underline{\text{finite}}\}$$

Remember P str. prime ideal if L/P is prime and nondegenerate.

Proposition 4

$$\{c_i\}_{i\in\mathbb{N}} \text{ infinite and } P \text{ maximal with respect to } P \cap \{c_i\}_{i\in\mathbb{N}} = \emptyset$$
$$\Downarrow$$
$$P \text{ is str. prime ideal}$$

(notice that each $c_{i+1} \in [\mathrm{Id}_L(c_i), \mathrm{Id}_L(c_i)]$)

Theorem 2

$$K(L) = \bigcap (\text{str. prime ideals of } L)$$

Corollary

L nondegenerate $\implies$ subdirect product of str. prime algebras.

References:

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