Locally finite dimenional simple Lie algebras in positive characteristic: interesting open problems.

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1 The classification of simple finite dimensional Lie algebras over algebraically closed fields of characteristic $p > 3$

Many of the present results and open problems in locally finite dimensional Lie algebra theory concern those algebras which are built out of simple finite dimensional algebras, the *locally simple Lie algebras*. There are many more simple finite dimensional algebras over fields of positive charcteristic than just the classical ones. In the first part of my talk I will introduce these. Roughly speaking, there are 3 different constructions to produce simple Lie algebras. Throughout this talk we assume that F **is an algebraically closed field of positive charcteristic** $p > 3$.

<u>The classical algebras</u>

We take the simple matrix Lie algebras and take as entries the elements of F:

$$\mathfrak{sl}(n,F), \quad \{A \in M(n,F) \mid \Lambda(Av,w) + \Lambda(v,Aw) = 0 \ \forall v,w \in F^n\},$$

where Λ is a nondegenerate form on F^n. The following Chevalley construction yields Lie algebras over arbitrary fields. Let L be a finite dimensional simple Lie algebra over $\mathbb{C}$ and H a CSA. There is a basis x_α of L such that the multiplication coefficients $C^\gamma_{\alpha,\beta}$ given by

$$[x_\alpha, x_\beta] = C^\gamma_{\alpha,\beta} x_\gamma$$

are integers of absolute value less than 5. The $\mathbb{Z}$-span $L_\mathbb{Z}$ of a Chevalley basis is a $\mathbb{Z}$-subalgebra in L. Then $L_F := L_\mathbb{Z} \otimes_\mathbb{Z} F$ is a Lie algebra over F, having a basis as above, the multiplication coefficients reduced $\mathrm{mod}(p)$. The algebra is simple except $L_F \cong A_l,\ l \equiv -1 \bmod (p)$. In this case L_F has a one-dimensional center $C = F(h_1 + 2h_2 + \cdots + lh_l)$. Then L_F/C is simple. These are the *simple classical* Lie algebras

$$A_n \, (p \not| n+1),\ \mathfrak{psl}(n+1)\,(p|n+1),\ B_n,\ C_n,\ D_n,\ G_2,\ F_4,\ E_6,\ E_7,\ E_8.$$

Note that, by abuse of the characteristic 0 notation, in characteristic p the class of simple classical Lie algebras **includes the exceptional types**. These Lie algebras are related to algebraic groups. They are restricted.

The graded Cartan type Lie algebras

Let L denote an arbitrary Lie algebra, $U(L)$ its universal enveloping algebra and $\Delta : U(L) \longrightarrow U(L) \otimes U(L)$ the coproduct in $U(L)$. The dual space

$$U(L)^* = \mathrm{Hom}_F(U(L), F)$$

becomes a commutative and associative algebra if one sets

$$(fg)(u) := (f \otimes g)(\Delta\, u),\ f \in U(L)^*,\ u \in U(L).$$

More precisely, $U(L)^*$ is a "divided power algebra". To describe such algebras we introduce the following notation.

If we are given some multi-indices $\underline{a}, \underline{b} \in \mathbb{N}^m$ then we write

- $\underline{a} \leq \underline{b}$ if and only if $a_i \leq b_i$ for all $i = 1, \ldots, m$;
- $\binom{\underline{a}}{\underline{b}} = \prod_{i=1}^m \binom{a_i}{b_i}$;
- $|\underline{a}| = \sum_{i=1}^m a_i$;
- $\tau(\underline{a}) = (p^{a_1} - 1, \ldots p^{a_m} - 1) \in \mathbb{N}^m$;
- $\varepsilon_i = (0, \ldots, 1, \ldots 0)$ with 1 in the i-th slot;
- $\underline{1} = (1, \ldots, 1)$

Define $\mathcal{O}((m))$ as the algebra of all formal sums

$$\mathcal{O}((m)) = \left\{ \sum_{\underline{a} \geq \underline{0}} \alpha(\underline{a}) x^{(\underline{a})} \mid \underline{a} \in \mathbb{N}^m,\ \alpha(\underline{a}) \in F \right\}$$

where the multiplication is given by the formula

$$\left(\sum_{\underline{a}} \alpha(\underline{a}) x^{(\underline{a})} \right) \left(\sum_{\underline{b}} \beta(\underline{b}) x^{(\underline{b})} \right) = \left(\sum_{\underline{c}} \gamma(\underline{c}) x^{(\underline{c})} \right)$$

with

$$\gamma(\underline{c}) = \sum_{\underline{0} \leq \underline{a} \leq \underline{c}} \alpha(\underline{a}) \beta(\underline{c} - \underline{a}) \binom{\underline{c}}{\underline{a}}.$$

If $\underline{n} \in \mathbb{N}^m$ then we set

$$\mathcal{O}(m; \underline{n}) = \left\{ \sum_{\underline{0} \leq \underline{a} \leq \tau(\underline{n})} \alpha(\underline{a}) x^{(\underline{a})} \right\}$$

and notice that thanks to the properties of binomial coefficients this subspace is a finite-dimensional subalgebra in $\mathcal{O}((m))$ of dimension $p^{|n|}$.

Next we define the "partial derivatives" $\partial_1, \ldots, \partial_n$ of the algebra $\mathcal{O}((m))$ defined by

$$\partial_i \left(\sum_{(\underline{a})} \alpha(\underline{a}) x^{(\underline{a})} \right) = \sum_{(\underline{a})} \alpha(\underline{a}) x^{(\underline{a}-\varepsilon_i)}.$$

Put

$$W((m)) := \left\{ \sum_{i=1}^{m} f_i \partial_i \mid f_i \in \mathcal{O}((m)) \right\},$$

and also for any tuple $\underline{n} \in \mathbb{N}^m$ we set

$$W(m;\underline{n}) := \left\{ \sum_{i=1}^{m} f_i \partial_i \mid f_i \in \mathcal{O}(m;\underline{n}) \right\}.$$

Now $W(m;\underline{n})$ is an algebra of derivations of $\mathcal{O}(m;\underline{n})$. In the particular case $n_1 = \ldots = n_m = 1$ we have that $\mathcal{O}(m;\underline{1})$ is isomorphic to the algebra of "truncated polynomials"

$$F[X_1, \ldots, X_m]/(X_1^p, \ldots, X_m^p)$$

and $W(m;\underline{1})$ is the restricted Lie algebra of all derivations of $\mathcal{O}(m;\underline{1})$.

The motivation for these constructions are quite natural: over $\mathbb{C}$ it makes sense to consider $x^{(\underline{a})} \sim \prod_{i=1}^{m} \frac{1}{a_i!} x_i^{a_i}$. Then $\mathcal{O}((m)) \cong \mathbb{C}[[x_1, \ldots, x_m]]$ is the algebra of power series, ∂_i is the ordinary i-th partial derivative and $W((m))$ is the Lie algebra of continuous derivations.

A) Witt algebras; type W.

The Lie algebras of the type $W(m;\underline{n})$ form one of four families of "graded Cartan type" algebras called *Witt algebras*.

The algebra $W(m;\underline{n})$ is a simple Lie algebra of dimension $mp^{|\underline{n}|}$. One obtains a $\mathbb{Z}$-grading of $W(m;\underline{n})$ if one sets

$$\deg(x^{(\underline{a})}\partial_i) = |\underline{a}| - 1.$$

Let $W(m;\underline{n})_i$ denote the space of homogeneous elements of degree i. Then the following relations take place:

$$W(m;\underline{n}) = \bigoplus_{i=-1}^{|\tau(\underline{n})|-1} W(m;\underline{n})_i,$$

$$W(m;\underline{n})_{-1} = \bigoplus_{i=1}^{m} F\partial_i,$$

$$W(m;\underline{n})_0 \cong \mathfrak{gl}(m),$$

$$W(m;\underline{n})_{|\tau(\underline{n})|-1} = \bigoplus_{i=1}^{m} F x^{(\tau(\underline{n}))}\partial_i.$$

The subalgebra

$$W(m;\underline{n})_{(0)} = \sum_{i\geq 0} W(m;\underline{n})_i$$

is the only proper subalgebra of minimal codimension. Therefore it is uniquely determined. As such it is invariant under all automorphisms of $W(m;\underline{n})$.

B) Special Lie algebras; type S.

For $m \geq 3$ we set

$$S(m;\underline{n}) = \left\{ \sum_{i=1}^{m} f_i\partial_i \in W(m;\underline{n}) \mid \sum_{i=1}^{m} \partial_i(f_i) = 0 \right\}.$$

The derived algebra $S(m;\underline{n})^{(1)}$ is a simple Lie algebra of dimension

$$\dim\, S(m;\underline{n})^{(1)} = (m-1)(p^{|\underline{n}|}-1).$$

If we denote by $D_{i,j}$ the mapping $\mathcal{O}((m)) \longrightarrow W((m))$ given by

$$D_{i,j}\left(\sum_{(\underline{a})} \alpha(\underline{a})x^{(\underline{a})}\right) = \sum_{(\underline{a})} \alpha(\underline{a})x^{(\underline{a}-\varepsilon_j)}\partial_i - \sum_{(\underline{a})} \alpha(\underline{a})x^{(\underline{a}-\varepsilon_i)}\partial_j$$

then

$$S(m;\underline{n})^{(1)} = \sum_{1\leq i<j\leq m} D_{i,j}(\mathcal{O}(m;\underline{n})).$$

The algebras $S(m;\underline{n})$ and $S(m;\underline{n})^{(1)}$ are graded subalgebras in $W(m;\underline{n})$. One has $S(m;\underline{n})^{(1)}{}_0 \cong \mathfrak{sl}(m)$.

C) Hamiltonian Lie algebras; type H.

In this section $m = 2r$. We define

$$j' = \begin{cases} j+r & 1\leq j\leq r, \\ j-r & r<j\leq 2r \end{cases} \qquad \sigma(j) = \begin{cases} 1, & 1\leq j\leq r, \\ -1, & r<j\leq 2r. \end{cases}$$

Now we set

$$H(2r;\underline{n}) = \left\{ \sum_{i=1}^{2r} f_i\partial_i \in W(2r;\underline{n}) \mid \sigma(j')\partial_i(f_{j'}) = \sigma(i')\partial_j(f_{i'}),\ 1\leq i,j\leq r \right\}.$$

We define a mapping $D_H : \mathcal{O}(2r;\underline{n}) \longrightarrow W(2r;\underline{n})$ by setting

$$D_H(f) = \sum_{i=1}^{2r} \sigma(j)\partial_j(f)\partial_{j'}, \quad f\in\mathcal{O}(2r;\underline{n}).$$

Then

$$H(2r;\underline{n})^{(2)} = \text{span}\,\{D_H(x^{(\underline{a})}) \mid \underline{0} < \underline{a} < \tau(\underline{n}) >$$

is a simple algebra. The algebras $H(2r;\underline{n})$, $H(2r;\underline{n})^{(1)}$ and $H(2r;\underline{n})^{(2)}$ are graded subalgebras in $W(2r;\underline{n})$. One has $H(m;\underline{n})^{(2)}{}_0 \cong \mathfrak{sp}(m)$.

D) Contact Lie algebras; type K.

Suppose $m = 2r+1 > 1$. We define $\sigma(j)$ and j' as in C). We also define $D_K : \mathcal{O}((m)) \longrightarrow W((m))$ by setting $D_K(f) = \sum_{i=1}^m f_i \partial_i$ where f_i is given by the equations

$$f_i = x^{(\varepsilon_i)}\partial_m(f) + \sigma(i')\partial_{i'}(f), \quad i \le 2r,$$
$$f_m = 2f - \sum_{i=1}^{2r} x^{(\varepsilon_i)}\partial_i(f).$$

Set

$$K(m;\underline{n}) = D_K(\mathcal{O}(m;\underline{n})).$$

Then $K(m;\underline{n})^{(1)}$ is a simple Lie algebra of dimension $p^{|\underline{n}|}$ if $m+3 \not\equiv 0 \pmod{(p)}$ and $p^{|\underline{n}|}-1$ otherwise. One imposes a grading on $K(m;\underline{n})$ by setting

$$\deg(D_K(x^{(\underline{a})})) = |\underline{a}| + a_m - 2.$$

E) Melikian algebras; type $\mathcal{M}$.

Given a Lie algebra M_0 and an M_0-module V the first Cartan prolongation of the pair (V, M_0) is

$$\mathcal{C}^{(1)}(V, M_0) := \{\varphi : V \to M_0 \mid \varphi(u)(v) = \varphi(v)(u) \ \forall\ u, v \in V\}.$$

If this first prolongation is nonzero it gives rise to infinite dimensional graded Lie algebras

$$\begin{array}{ccccccc} V & \oplus & M_0 & \oplus & \mathcal{C}^{(1)}(V,M_0) & & \\ \| & & \| & & \cup & & \\ M_{-1} & \oplus & M_0 & \oplus & M_1 & & \longrightarrow \end{array}$$

These are freely generated modulo the Lie relations given naturally by these 3 spaces. We are interested in homomorphic images of this Lie algebra of finite dimension. This is the way Cartan proceeded to classify certain classes of infinite dimensional Lie algebras having a filtration of depth 1 (i.e., $L = L_{(-1)}$). More generally, Kac investigates Lie algebras freely generated (modulo the canonical relations) by "local Lie algebras"

$$\longleftarrow \quad M_{-1} \quad \oplus \quad M_0 \quad \oplus \quad M_1 \quad \longrightarrow .$$

Just by chance, as $p = 5$, there are finite dimensional homomorphic images different from classical or Cartan type, which are simple Lie algebras. They have a grading of depth 3. This way the 2-parameter family of *Melikian algebras* $\mathcal{M}(n_1, n_2)$ of dimension $5^{n_1+n_2+1}$ occur. As a vector space with some subtle multiplication table it has the form

$$\mathcal{M}(n_1, n_2) = W(2; (n_1, n_2)) \oplus W(2; (n_1, n_2)) \oplus \mathcal{O}(2; (n_1, n_2)).$$

The filtered Lie algebras of Cartan type

Any grading $L = \bigoplus_{i \in \mathbb{Z}} L_i$ defines a filtration on L if one sets $L_{(j)} = \sum_{j \leq i} L_i$. Let L be a Cartan type Lie algebra with a grading as mentioned before, or Melikian with the (not explicitely described) grading. It turns out that the respective subalgebra $L_{(0)}$ is uniquely described as the only proper subalgebra of minimal codimension (in case of Cartan type algebras) and with the additional property that the filtration coming with it has depth 3.

We will be saying that $L_{(0)} = \sum_{i \geq 0} L_i$ is the *natural* maximal subalgebra in L. Now $L_{(1)}$ is the unique maximal ideal in $L_{(0)}$ whose action on L is nilpotent. Besides,

$$\begin{aligned} L_{(-1)} &= \{x \in L \mid [x, L_{(1)}] \subset L_{(0)}\}, \\ L_{(i+1)} &= \{x \in L_{(i)} | [x, L_{(-1)}] \subset L_{(i)}\} \text{ for } i \geq 0, \\ L_{(i-1)} &= [L_{(i)}, L_{(-1)}] + L_{(i)} \text{ for } i < 0, \end{aligned}$$

so that $L_{(0)}$ uniquely determines this *natural* filtration. The length of the filtration

$$L = L_{(-r)} \supset L_{(-r+1)} \supset \ldots \supset L_{(s)} \supset \{0\}$$

can be given by the following

	$W(m; \underline{n})$	$S(m; \underline{n})^{(1)}$	$H(m; \underline{n})^{(2)}$	$K(m; \underline{n})^{(1)}$	$\mathcal{M}(n_1, n_2)$
r	1	1	1	2	3
s	$\lvert\tau(\underline{n})\rvert - 1$	$\lvert\tau(\underline{n})\rvert - 2$	$\lvert\tau(\underline{n})\rvert - 3$	$\lVert\tau(\underline{n})\rVert, m+3 \not\equiv 0 (\operatorname{mod} p)$, $\lVert\tau(\underline{n})\rVert - 1, m+3 \equiv 0 (\operatorname{mod} p)$	$3(5^{n_1} + 5^{n_2}) - 7$

The following definition describes the filtered Lie algebras of Cartan type.

Definition 1 *Let L be a simple Lie algebra with filtration $L = L_{(-r)} \supset \cdots \supset L_{(s)} \supset \{0\}$. If there exists $\mathcal{X} \in \{W, S, H, K\}$ and an embedding $\psi : \operatorname{gr} L \hookrightarrow \mathcal{X}(m; \underline{n})$ of graded algebras, such that*

$$\mathcal{X}(m; \underline{n})^{(2)} \subset \psi(\operatorname{gr} L) \subset \mathcal{X}(m; \underline{n}),$$

then L is called a simple Cartan type Lie algebra of the type $\mathcal{X}$.

The compatibility property

The monomials $x^{(\underline{a})}$ are connected with a sequence of continuous mappings $u \to u^{(j)}$, $(j \geq 0)$ from $\mathcal{O}((m))_{(1)}$ into $\mathcal{O}((m))$ satisfying the conditions

$$\begin{aligned} u^{(0)} &= 1, \quad u^{(1)} = u, \\ (u+v)^{(j)} &= \sum_{i=0}^{j} u^{(i)} v^{(j-i)}, \\ u^{(i)} u^{(j)} &= \binom{i+j}{i} u^{(i+j)}, \\ (u^{(i)})^{(j)} &= ((ij)!/(i!)^j j!) u^{(ij)}, \\ (uv)^{(j)} &= u^j v^{(j)} \end{aligned}$$

for all $u, v \in \mathcal{O}((m))_{(1)}$. Then $x^{(\underline{a})} = \prod_{i=1}^{m} x_i^{(a_i)}$.

Let $\mathrm{Aut}_c\, \mathcal{O}((m))$ denote the group of continuous automorphisms of the algebra $\mathcal{O}((m))$ each of its elements ϕ satisfying the additional condition

$$\phi(u^{(j)}) = (\phi(u))^{(j)} \quad \forall\, u \in \mathcal{O}((m))_{(1)},\ j \geq 1.$$

Each of the automorphism $\phi \in \mathrm{Aut}_c\, \mathcal{O}((m))$ leaves $W((m))$ invariant together with its filtration, i.e.

$$\phi \circ W((m)) \circ \phi^{-1} \subset W((m)), \quad \phi \circ W((m))_{(i)} \circ \phi^{-1} \subset W((m))_{(i)} \quad \text{for all } i.$$

Definition 2 *For any $\phi \in \mathrm{Aut}_c\, \mathcal{O}((m))$ and $\mathcal{X} = W,\ S,\ H$ or K we set*

$$\mathcal{X}(m; \underline{n}; \phi) := \phi \circ \mathcal{X}((m)) \circ \phi^{-1} \cap W(m; \underline{n}).$$

We have

Theorem 1 (The compatibility property) *Every simple Lie algebra of Cartan type may be viewed as*

$$L = \mathcal{X}(m; \underline{n}; \phi)^{(2)}, \quad \mathcal{X}(m; \underline{n})^{(2)} \subset \mathrm{gr}\, L \subset \mathcal{X}(m; \underline{n}).$$

Here $\mathcal{X}$ and m are uniquely determined and $\underline{n}$ is determined up to some permutation of indices. □

In the case of this theorem one has

$$\mathcal{X}(m; \underline{n})_{(i)} \subset \mathcal{X}(m; \underline{n}; \phi)^{(2)}{}_{(i)} + W(m; \underline{n})_{(i+1)},\ i \leq p-3.$$

As a result we obtain that
- $\mathcal{X}(m; \underline{n}; \phi)^{(2)}{}_{(0)}$ is the unique subalgebra of minimal codimension,
- $\mathcal{X}(m; \underline{n}; \phi)^{(2)}{}_{(1)}$ is the unique maximal ideal in $\mathcal{X}(m; \underline{n}; \phi)^{(2)}{}_{(0)}$ with nilpotent

action on $\mathcal{X}(m;\underline{n};\phi)^{(2)}$ and
- $\mathcal{X}(m;\underline{n};\phi)^{(2)}{}_{(-1)}/\mathcal{X}(m;\underline{n};\phi)^{(2)}{}_{(0)}$ is the unique irreducible $\mathcal{X}(m;\underline{n};\phi)^{(2)}{}_{(0)}$-module.

This compatibility property means that the filtered Lie algebras of Cartan type are filtered deformations of graded Cartan type algebras inside some Witt algebras. These deformations are described by some cohomology groups. As a result, every Witt algebra, Contact algebra or Melikian algebra is isomorphic to its underlying graded algebra. Only the Special and Hamiltonian algebras allow filtered deformations.

The final classification theorem at present is

Theorem 2 (Block-Premet-Strade-Wilson 2004) *Suppose F is an algebraically closed field of characteristic $p > 3$ and L is a finite dimensional simple Lie algebra over F. Then L is of classical, Cartan, or Melikian type.* □

2 Two early results on locally finite dimensional Lie algebras

In 1994 Bahturin and I published a paper "Locally finite-dimensional simple Lie algebras". That paper resulted from a visit of Bahturin in Hamburg and seems to be the first paper at all on that subject. In one of the theorems we used the classification of finite dimensional simple Lie algebras in the status of that time and proved

Theorem 3 *Let L be a simple infinite dimensional locally finite dimensional Lie algebra over an algebraically closed field of characteristic $p > 7$. Suppose that there exists $d = d(L) \in \mathbb{N}$ with the following property:*

- if U is a finite dimensional subalgebra in L and I is a maximal ideal of U with a classical factor U/I, then $\dim U/I \leq d$.

Then there are a local system (L_i) of L, $\mathcal{X} \in \{W, S, H, K\}$, $m \in \mathbb{N}$, a sequence $\underline{n}^1 \leq \underline{n}^2 \ldots$ of m-tuples, and automorphisms $\varphi^i \in \mathrm{Aut}_c\,\mathcal{O}((m))$ such that $L_i \cong \mathcal{X}(m;\underline{n}^i;\varphi^i)^{(2)}$ for all i. □

One can easily extend this to the case $p \geq 5$.

In another attempt we constructed a direct limit of Witt algebras, where quite opposite to the previous theorem the 0-components grow but not the exponents of the monomials.

Theorem 4 *Let $2m_1 \leq m_2$ and $\rho : \mathfrak{gl}(m_1) \to \mathfrak{gl}(m_2 - m_1)$ denote an embedding. Identify*

$$\mathfrak{gl}(m_1) \cong \sum_{i.j=1}^{m_1} F x_i \partial_j \text{ and } \mathfrak{gl}(m_2 - m_1) \cong \sum_{i.j=m_1+1}^{m_2} F x_i \partial_j.$$

The following formula defines an embedding $\bar{\rho} : W(m_1; \underline{1}) \to W(m_2; \underline{1})$

$$\bar{\rho}(x^{(\underline{a})}\partial_i) := x^{(\underline{a})}\partial_i + \sum_{s=1}^{m_1} x^{(\underline{a}-\epsilon_s)}\rho(x_s\partial_i).$$

Restricting $\bar{\rho}$ to the homogeneous degree 0 component gives an embedding $\bar{\rho}_0 : \mathfrak{gl}(m_1) \to \mathfrak{gl}(m_2)$. Then $\bar{\rho}$ extends the embedding $\bar{\rho}_0$.

3 Problems

In the context of Theorems 3 and 4 many problems arise which we have not attacked at that time.

(1) <u>The graded case:</u> (managable)

Let $\mathcal{X} \in \{W, S, H, K, \mathcal{M}\}$, $\underline{n}^1 \leq \underline{n}^2 \ldots$ be m-tuples, $\mathcal{X}(m; \underline{n}^1)^{(2)} \subset \mathcal{X}(m; \underline{n}^2)^{(2)} \cdots$ with the natural embedding induced by the inclusions

$$\mathcal{O}(m; \underline{n}^1) \subset \mathcal{O}(m; \underline{n}^2) \cdots$$

and $L = \mathcal{X}(m; \underline{n})$ with $\underline{n} \in (\mathbb{N} \cup \{\infty\})^m$ the direct limit algebra.

(a) Does there exist a unique maximal subalgebra $\mathcal{X}(m; \underline{n})_{(0)}$ of minimal codimension? How can one characterize this invariantly? Is this a restricted subalgebra? A positive answer describes $\mathcal{X}$ and m invariantly.

(b) Under which conditions do sandwich elements exist? For instance, if $(n_j^i)_{i>0}$ are bounded for some indices j? If so, can $\mathcal{X}(m; \underline{n})_{(0)}$ be characterized by sandwich elements?

(c) There are many publications on compatible root space decompositions, mostly for direct limits of classical algebras by Penkov and others. Are there compatible root space decompositions for suitable maximal tori in the present cases? For example, the torus $(\sum_{j=1}^m Fx_j\partial_j) \cap \mathcal{X}(m; \underline{n})$ should behave fine. What results in charcteric 0 have analogues here?

(2) <u>The non-graded case:</u> (challenging)

Let L have a local system (L_i) such that there are $\mathcal{X} \in \{S, H\}$, $m \in \mathbb{N}$, a sequence $\underline{n}^1 \leq \underline{n}^2 \ldots$ of m-tuples, and automorphisms $\varphi^i \in \mathrm{Aut}_c\, \mathcal{O}((m))$ such that $\psi_i : L_i \cong \mathcal{X}(m; \underline{n}^i; \varphi^i)^{(2)}$ for all i. Suppose that the embeddings

$$\begin{array}{ccc} L_i & \hookrightarrow & L_{i+1} \\ \downarrow\ \psi_i & & \downarrow\ \psi_i \\ \mathcal{X}(m; \underline{n}^i; \varphi^i)^{(2)} & \hookrightarrow & \mathcal{X}(m; \underline{n}^{i+1}; \varphi^{i+1})^{(2)} \end{array}$$

and the isomorphisms ψ_i respect the natural filtrations of the Cartan type algebras.

(a) What embeddings $\mathcal{X}(m;\underline{n}^i;\varphi^i)^{(2)} \hookrightarrow \mathcal{X}(m;\underline{n}^{i+1};\varphi^{i+1})^{(2)}$ can occur? Every such embedding gives an embedding of the associated graded algebras. In particular, one would like to answer this question for the graded case.

(b) What are the filtered deformations of the algebras obtained by the process in (1) by using graded algebras? Here one has to compute some low dimensional cohomology groups. Which of these deformations yield locally finite dimensional algebras?

(c) What does the compatibility property mean in this context?

(d) Investigate the problems (1a) - (1c) for the non-graded case.

(3) Extending direct limits of classical algebras: (quite interesting)

Let $W((\rho);\underline{1})$ denote the direct limit for a family of embeddings $\rho_i : \mathfrak{gl}(m_i) \to \mathfrak{gl}(m_{i+1})$ as in Theorem 4.

(a) Are there respective extensions for $\big(\mathfrak{sl}(m_i)\big)$ and $\big(S(m_i;\underline{1})^{(1)}\big)$, and $\mathfrak{sp}(m_i)$ and $\big(H(m_i;\underline{1})^{(2)}\big)$, respectively?

(b) Show that every diagonal embedding of $\mathfrak{gl}$'s is obtained as a suitable $\bar{\rho}_0$. More generally, can one characterize the embeddings obtained as a suitable $\bar{\rho}_0$?

(c) Does $\bar{\rho}$ respect sandwich elements?

(d) Is $W((\rho);\underline{1})$ restricted?

(e) Is it possible to characterize the direct limit of the natural maximal subalgebras by internal properties? If so, can one rediscover the original embeddings ρ_i?

(f) Is it possible to characterize those direct limits of Witt algebras which arise from diagonal embeddings and classify the extensions as well?

(g) What about finitary algebras and their extensions to Cartan type algebras?

(h) If the original embeddings are root reductive, are the extension so as well?