On Homomorphisms of Dialgonal Lie Algebras

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- Homomorphisms of diagonal Lie algebras
Basic definitions

The base field is $\mathbb{C}$. All Lie algebras considered are finite dimensional or countable dimensional.

**Definition.** A Lie algebra $\mathfrak{g}$ is *locally finite* if any finite subset $S$ of $\mathfrak{g}$ is contained in a finite-dimensional Lie subalgebra $\mathfrak{g}(S)$ of $\mathfrak{g}$. If, for any $S$, $\mathfrak{g}(S)$ can be chosen simple, $\mathfrak{g}$ is called *locally simple*.

**Definition.** An *exhaustion* $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots$ of a locally finite Lie algebra $\mathfrak{g}$ is a direct system of finite-dimensional Lie subalgebras of $\mathfrak{g}$ such that the direct limit Lie algebra $\lim_{\to} \mathfrak{g}_n$ is isomorphic to $\mathfrak{g}$. 
Basic definitions

**Definition.** An injective homomorphism $\varepsilon : g_1 \rightarrow g_2$ of finite-dimensional classical simple Lie algebras is called *diagonal* if there is an isomorphism of $g_1$-modules

$$V_2 \downarrow g_1 \cong V_1 \oplus \ldots \oplus V_1 \oplus V_1^* \oplus \ldots \oplus V_1^* \oplus T_1 \oplus \ldots \oplus T_1,$$

where $V_i$ is the natural $g_i$-module ($i = 1, 2$), $V_1^*$ is the dual of $V_1$, and $T_1$ is the one-dimensional trivial $g_1$-module. The triple $(l, r, z)$ is called the *signature* of $\varepsilon$.

**Definition.** A locally simple Lie algebra $s$ is *diagonal* if it admits an exhaustion by simple subalgebras $s_i$ such that all inclusions $s_i \subset s_{i+1}$ are diagonal.
Origin of the problems

- Classification of pairs $\mathfrak{s} \subset \mathfrak{g}$ of finite-dimensional semisimple Lie algebras up to $\mathfrak{g}$-conjugacy (Malcev, Dynkin).
  For classical $\mathfrak{s}, \mathfrak{g}$: the study of $\mathfrak{g}$-conjugacy classes of $\mathfrak{s}$ is equivalent to the study of $V \downarrow \mathfrak{s}$ ($V$ is the natural $\mathfrak{g}$-module).

- Description of locally semisimple Lie subalgebras of $\mathfrak{g} \cong \mathfrak{gl}(\infty), \mathfrak{sl}(\infty), \mathfrak{so}(\infty), \mathfrak{sp}(\infty)$ up to isomorphism.
  Description of $V \downarrow \mathfrak{s}$ and $V^* \downarrow \mathfrak{s}$ in terms of the socle filtration (Dimitrov, Penkov).

- Classification of diagonal locally simple Lie algebras up to isomorphism (Baranov, Zhilinskii).
Let $\mathfrak{s} = \bigcup_i \mathfrak{s}_i$ be an infinite-dimensional diagonal Lie algebra. The triple $(l_i, r_i, z_i)$ denotes the signature of the homomorphism $\mathfrak{s}_i \to \mathfrak{s}_{i+1}$ and $n_i$ denotes the dimension of the natural $\mathfrak{s}_i$-module.

We can assume that

- all $\mathfrak{s}_i$ are of the same type $X$ ($X = A$, $C$, or $O$);
- $r_i = 0$ if $X$ is not $A$ and $l_i \geq r_i$ if $X = A$;
- $n_1 = 1$, $l_1 = n_2$, $r_1 = z_1 = 0$.

We will write $\mathfrak{s} = X(\mathcal{T})$, where $\mathcal{T} = \{(l_i, r_i, z_i)\}_{i \in \mathbb{N}}$. 
Diagonal Lie algebras

Set $s_i = l_i + r_i$, $c_i = l_i - r_i$, $S = (s_i)_{i \in \mathbb{N}}$, $C = (c_i)_{i \in \mathbb{N}}$. Then $Stz(S) = s_1 s_2 \cdots$ and $Stz(C) = c_1 c_2 \cdots$.

Put $\delta_i = \frac{s_1 \cdots s_{n-1}}{n_i}$ and $\sigma_i = \frac{c_1 \cdots c_i}{s_1 \cdots s_i}$. The limit $\delta(T) = \lim_{i \to \infty} \delta_i$ is called the density index of $T$ and the limit $\sigma(T) = \lim_{i \to \infty} \sigma_i$ is called the symmetry index of $T$.

Density types of $T$:
- $T$ is pure, if $\delta_i = \delta_{i_0} > 0$ for all $i > i_0$;
- $T$ is dense, if $0 < \delta < \delta_i$ for all $i$;
- $T$ is sparse, if $\delta = 0$.

Symmetry types of $T$:
- $T$ is one-sided, if $c_i = s_i$ for all $i \geq i_0$;
- $T$ is two-sided symmetric, if there exist infinitely many $c_i = 0$;
- $T$ is two-sided weakly non-symmetric, if $\sigma(T) = 0$;
- $T$ is two-sided strongly non-symmetric, if $\sigma(T) > 0$. 
Diagonal Lie algebras

**Theorem** (Baranov, Zhilinskii) Let $X = A$, $C$, or $O$ and let $\mathcal{T} = \{(l_i, r_i, z_i)\}$. Then $X(\mathcal{T}) \cong X(\mathcal{T}')$ if and only if the following conditions hold.

(A₁) The sequences $\mathcal{T}$ and $\mathcal{T}'$ have the same density type.

(A₂) $Stz(S) \cong_{Q} Stz(S')$.

(A₃) $\frac{\delta}{\delta'} \in \frac{Stz(S)}{Stz(S')}$ for dense and pure sequences.

(B₁) The sequences $\mathcal{T}$ and $\mathcal{T}'$ have the same symmetry type.

(B₂) $Stz(C) \cong_{Q} Stz(C')$ for two-sided non-symmetric sequences.

(B₃) There exists $\alpha \in \frac{Stz(S)}{Stz(S')}$ such that $\alpha \frac{\sigma}{\sigma'} \in \frac{Stz(C)}{Stz(C')}$ for two-sided strongly non-symmetric sequences. Moreover, $\alpha = \frac{\delta}{\delta'}$ if in addition the triple sequences are dense or pure.
Theorem (Baranov, Zhilinskii) Let $\mathcal{T} = \{(l_i, r_i, z_i)\}$, $\mathcal{T}' = \{(l'_i, 0, z'_i)\}$, and $\mathcal{T}'' = \{(l''_i, 0, z''_i)\}$.

(i) $A(\mathcal{T}) \cong O(\mathcal{T}')$ (resp., $A(\mathcal{T}) \cong C(\mathcal{T}')$) if and only if $\mathcal{T}$ is two-sided symmetric, $2^\infty$ divides $\text{Stz}(S')$, and the conditions $(A_1)$, $(A_2)$, $(A_3)$ hold.

(ii) $O(\mathcal{T}') \cong C(\mathcal{T}'')$ if and only if $2^\infty$ divides both $\text{Stz}(S')$, and $\text{Stz}(S'')$, and the conditions $(A_1)$, $(A_2)$, $(A_3)$ hold.
The statement of the theorem

**Theorem (M)**

a) The three finitary Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{so}(\infty)$, $\mathfrak{sp}(\infty)$ admit an injective homomorphism into any infinite-dimensional diagonal Lie algebra. An infinite-dimensional non-finitary diagonal Lie algebra admits no injective homomorphism into $\mathfrak{sl}(\infty)$, $\mathfrak{so}(\infty)$, $\mathfrak{sp}(\infty)$.

b) Let $\mathfrak{s}_1 = X_1(T_1)$, $\mathfrak{s}_2 = X_2(T_2)$ be infinite-dimensional non-finitary diagonal Lie algebras. Set $S_i = \text{Stz}(S_i)$, $S = \text{GCD}(S_1, S_2)$, $R_i = \frac{S_i}{S}$, $\delta_i = \delta(T_i)$, $C_i = \text{Stz}(C_i)$, $C = \text{GCD}(C_1, C_2)$, $B_i = \frac{C_i}{C}$, and $\sigma_i = \sigma(T_i)$ for $i = 1, 2$. Then $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$ if and only if the following conditions hold.

1) $R_1$ is finite.

2) $\mathfrak{s}_2$ is sparse if $\mathfrak{s}_1$ is sparse.
3) If $s_1$ and $s_2$ are non-sparse, both $R_1$ and $R_2$ are finite, and $S$ is not divisible by an infinite power of any prime number, then $\epsilon \frac{R_1}{\delta_1} \leq \frac{R_2}{\delta_2}$ for $\epsilon$ as specified below. The inequality is strict if $s_1$ is pure and $s_2$ is dense. We have $\epsilon = 2$, except in the cases listed below, in which $\epsilon = 1$:

3.1) $(X_1, X_2) = (C, C), (O, O), (C, A), (O, A),$ and $(X_1, X_2) = (A, A)$ with both $s_1$ and $s_2$ being one-sided;

3.2) $(X_1, X_2) = (A, A), B_1$ is finite, either $s_1$ is one-sided and $s_2$ is two-sided non-symmetric or $s_2$ is two-sided weakly non-symmetric and $s_1$ is two-sided non-symmetric;

3.3) $(X_1, X_2) = (A, A), B_1$ is finite, both $s_1$ and $s_2$ are two-sided strongly non-symmetric, either $B_2$ is infinite or $C$ is divisible by an infinite power of any prime number;

3.4) $(X_1, X_2) = (A, A), \text{ both } B_1$ and $B_2$ are finite, both $s_1$ and $s_2$ are two-sided strongly non-symmetric, $C$ is not divisible by an infinite power of any prime number, and $\frac{R_1 \sigma_1}{B_1} \geq \frac{R_2 \sigma_2}{B_2}$.
Ideas of the proof \((\mathfrak{sl}(\infty) \rightarrow \text{pure one-sided})\)

\[
\begin{array}{cccccc}
\mathfrak{sl}(2) & \rightarrow & \cdots & \rightarrow & \mathfrak{sl}(k) & \rightarrow \mathfrak{sl}(k+1) & \rightarrow \cdots \\
\theta_2 \downarrow & & & \theta_k \downarrow & & \theta_{k+1} \downarrow & \\
\mathfrak{sl}(n_1 n_2) & \rightarrow & \cdots & \rightarrow & \mathfrak{sl}(n_1 \cdots n_k) & \rightarrow \mathfrak{sl}(n_1 \cdots n_{k+1}) & \rightarrow \cdots \\
\end{array}
\]

We choose \(\theta_k\) such that as \(\mathfrak{sl}(k)\)-modules

\[V_k \downarrow \mathfrak{sl}(k) \cong a_0^k \wedge^0 (F_k) \oplus a_1^k \wedge^1 (F_k) \oplus \cdots \oplus a_k^k \wedge^k (F_k).\]

\[
\begin{align*}
a_0^0 & \\
a_0^1 & a_1^1 \\
a_0^2 & a_1^2 & a_2^2 \\
\cdots & \\
\end{align*}
\]

with the conditions

\[a_i^k + a_{i+1}^k = n_k a_i^{k-1}, \ k \geq 1 \text{ and } a_0^0 = 1.\]
Ideas of the proof (sparse one-sided $\not\rightarrow$ pure one-sided)

\[ \text{sl}(n_1) \rightarrow \cdots \rightarrow \text{sl}(n_i) \rightarrow \text{sl}(n_{i+1}) \rightarrow \cdots \]
\[ \theta_1 \downarrow \quad \theta_i \downarrow \quad \theta_{i+1} \downarrow \]
\[ \text{sl}(m_1) \rightarrow \cdots \rightarrow \text{sl}(m_1 \cdots m_i) \rightarrow \text{sl}(m_1 \cdots m_{i+1}) \rightarrow \cdots \]

\[ V_i \downarrow \text{sl}(n_i) \cong \bigoplus_{\lambda \in H_i} F_{n_i}^\lambda \bigoplus \cdots \bigoplus F_{n_i}^\lambda, \quad d_i = \max_{\lambda \in H_i} (\lambda_1 - \lambda n_i). \]

Using branching rules we prove that $d_i \geq d_{i+1}$. Denote $d = \lim d_i$.

Then $\lambda_{d+1} = \lambda_{d+2} = \cdots = \lambda_{n_i-d}$ for $\lambda \in H_i$ for large enough $i$, which yields $I(\theta_i) \leq \frac{c_0 n_i}{n_i^2 - 1} m_1 \cdots m_i$.

By calculating $I_{\text{sl}(m_1 \cdots m_i)}^{\text{sl}(n_1)}$ in two ways we get $\frac{l_1 \cdots l_{i-1}}{n_i} \geq c$. 
Diagonal and non-diagonal homomorphisms

Natural representations of diagonal Lie algebras

A natural $g$-module is any non-zero $g$-module which can be constructed as a direct limit $V = \lim_{\rightarrow} V_n$, where $V_n$ is the natural $g_n$-module.

Inductive systems
The end

Thank you for your attention!