On Homomorphisms of Dialgonal Lie Algebras

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Basic definitions

The base field is $\mathbb{C}$. All Lie algebras considered are finite dimensional or countable dimensional.

Definition. A Lie algebra $\mathfrak{g}$ is *locally finite* if any finite subset S of $\mathfrak{g}$ is contained in a finite-dimensional Lie subalgebra $\mathfrak{g}(S)$ of $\mathfrak{g}$. If, for any S, $\mathfrak{g}(S)$ can be chosen simple, $\mathfrak{g}$ is called *locally simple*.

Definition. An *exhaustion*

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots$$

of a locally finite Lie algebra $\mathfrak{g}$ is a direct system of finite-dimensional Lie subalgebras of $\mathfrak{g}$ such that the direct limit Lie algebra $\varinjlim \mathfrak{g}_n$ is isomorphic to $\mathfrak{g}$.

Basic definitions

Definition. An injective homomorphism $\varepsilon : \mathfrak{g}_1 \to \mathfrak{g}_2$ of finite-dimensional classical simple Lie algebras is called *diagonal* if there is an isomorphism of $\mathfrak{g}_1$-modules

$$V_2 \downarrow \mathfrak{g}_1 \cong \underbrace{V_1 \oplus \ldots \oplus V_1}_{l} \oplus \underbrace{V_1^* \oplus \ldots \oplus V_1^*}_{r} \oplus \underbrace{T_1 \oplus \ldots \oplus T_1}_{z},$$

where V_i is the natural $\mathfrak{g}_i$-module $(i = 1, 2)$, V_1^* is the dual of V_1, and T_1 is the one-dimensional trivial $\mathfrak{g}_1$-module. The triple (l, r, z) is called the *signature* of ε.

Definition. A locally simple Lie algebra $\mathfrak{s}$ is *diagonal* if it admits an exhaustion by simple subalgebras $\mathfrak{s}_i$ such that all inclusions $\mathfrak{s}_i \subset \mathfrak{s}_{i+1}$ are diagonal.

Origin of the problems

- Classification of pairs $\mathfrak{s} \subset \mathfrak{g}$ of finite-dimensional semisimple Lie algebras up to $\mathfrak{g}$-conjugacy (Malcev, Dynkin).
  For classical $\mathfrak{s}, \mathfrak{g}$: the study of $\mathfrak{g}$-conjugacy classes of $\mathfrak{s}$ is equivalent to the study of $V \downarrow \mathfrak{s}$ (V is the natural $\mathfrak{g}$-module).
- Description of locally semisimple Lie subalgebras of $\mathfrak{g} \cong \mathrm{gl}(\infty), \mathrm{sl}(\infty), \mathrm{so}(\infty), \mathrm{sp}(\infty)$ up to isomorphism.
  Description of $V \downarrow \mathfrak{s}$ and $V^* \downarrow \mathfrak{s}$ in terms of the socle filtration (Dimitrov, Penkov).
- Classification of diagonal locally simple Lie algebras up to isomorphism (Baranov, Zhilinskii).

Diagonal Lie algebras

Let $\mathfrak{s} = \cup_i \mathfrak{s}_i$ be an infinite-dimensional diagonal Lie algebra.

The triple (l_i, r_i, z_i) denotes the signature of the homomorphism $\mathfrak{s}_i \to \mathfrak{s}_{i+1}$ and n_i denotes the dimension of the natural $\mathfrak{s}_i$-module.

We can assume that

- all $\mathfrak{s}_i$ are of the same type X ($X = A$, C, or O);
- $r_i = 0$ if X is not A and $l_i \geq r_i$ if $X = A$;
- $n_1 = 1$, $l_1 = n_2$, $r_1 = z_1 = 0$.

We will write $\mathfrak{s} = X(\mathcal{T})$, where $\mathcal{T} = \{(l_i, r_i, z_i)\}_{i \in \mathbb{N}}$.

Diagonal Lie algebras

Set $s_i = l_i + r_i$, $c_i = l_i - r_i$, $\mathcal{S} = (s_i)_{i\in\mathbb{N}}$, $\mathcal{C} = (c_i)_{i\in\mathbb{N}}$. Then $\mathrm{Stz}(\mathcal{S}) = s_1 s_2 \cdots$ and $\mathrm{Stz}(\mathcal{C}) = c_1 c_2 \cdots$.

Put $\delta_i = \frac{s_1 \cdots s_{n-1}}{n_i}$ and $\sigma_i = \frac{c_1 \cdots c_i}{s_1 \cdots s_i}$. The limit $\delta(\mathcal{T}) = \lim\limits_{i\to\infty} \delta_i$ is called the *density index* of $\mathcal{T}$ and the limit $\sigma(\mathcal{T}) = \lim\limits_{i\to\infty} \sigma_i$ is called the *symmetry index* of $\mathcal{T}$.

Density types of $\mathcal{T}$:

- $\mathcal{T}$ is *pure*, if $\delta_i = \delta_{i_0} > 0$ for all $i > i_0$;
- $\mathcal{T}$ is *dense*, if $0 < \delta < \delta_i$ for all i;
- $\mathcal{T}$ is *sparse*, if $\delta = 0$.

Symmetry types of $\mathcal{T}$:

- $\mathcal{T}$ is *one-sided*, if $c_i = s_i$ for all $i \geq i_0$;
- $\mathcal{T}$ is *two-sided symmetric*, if there exist infinitely many $c_i = 0$;
- $\mathcal{T}$ is *two-sided weakly non-symmetric*, if $\sigma(\mathcal{T}) = 0$;
- $\mathcal{T}$ is *two-sided strongly non-symmetric*, if $\sigma(\mathcal{T}) > 0$.

Diagonal Lie algebras

Theorem (Baranov, Zhilinskii) Let $X = A$, C, or O and let $\mathcal{T} = \{(l_i, r_i, z_i)\}$. Then $X(\mathcal{T}) \cong X(\mathcal{T}')$ if and only if the following conditions hold.

($\mathcal{A}_1$) The sequences $\mathcal{T}$ and $\mathcal{T}'$ have the same density type.

($\mathcal{A}_2$) $\mathrm{Stz}(\mathcal{S}) \overset{\mathbb{Q}}{\sim} \mathrm{Stz}(\mathcal{S}')$.

($\mathcal{A}_3$) $\frac{\delta}{\delta'} \in \frac{\mathrm{Stz}(\mathcal{S})}{\mathrm{Stz}(\mathcal{S}')}$ for dense and pure sequences.

($\mathcal{B}_1$) The sequences $\mathcal{T}$ and $\mathcal{T}'$ have the same symmetry type.

($\mathcal{B}_2$) $\mathrm{Stz}(\mathcal{C}) \overset{\mathbb{Q}}{\sim} \mathrm{Stz}(\mathcal{C}')$ for two-sided non-symmetric sequences.

($\mathcal{B}_3$) There exists $\alpha \in \frac{\mathrm{Stz}(\mathcal{S})}{\mathrm{Stz}(\mathcal{S}')}$ such that $\alpha\frac{\sigma}{\sigma'} \in \frac{\mathrm{Stz}(\mathcal{C})}{\mathrm{Stz}(\mathcal{C}')}$ for two-sided strongly non-symmetric sequences. Moreover, $\alpha = \frac{\delta}{\delta'}$ if in addition the triple sequences are dense or pure.

Diagonal Lie algebras

Theorem (Baranov, Zhilinskii) Let $\mathcal{T} = \{(l_i, r_i, z_i)\}$, $\mathcal{T}' = \{(l'_i, 0, z'_i)\}$, and $\mathcal{T}'' = \{(l''_i, 0, z''_i)\}$.

(i) $A(\mathcal{T}) \cong O(\mathcal{T}')$ (resp., $A(\mathcal{T}) \cong C(\mathcal{T}')$) if and only if $\mathcal{T}$ is two-sided symmetric, 2^∞ divides $\mathrm{Stz}(\mathcal{S}')$, and the conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$ hold.

(ii) $O(\mathcal{T}') \cong C(\mathcal{T}'')$ if and only if 2^∞ divides both $\mathrm{Stz}(\mathcal{S}')$, and $\mathrm{Stz}(\mathcal{S}'')$, and the conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$ hold.

The statement of the theorem

Theorem (M)

a) The three finitary Lie algebras $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, $\mathrm{sp}(\infty)$ admit an injective homomorphism into any infinite-dimensional diagonal Lie algebra. An infinite-dimensional non-finitary diagonal Lie algebra admits no injective homomorphism into $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, $\mathrm{sp}(\infty)$.

b) Let $\mathfrak{s}_1 = X_1(\mathcal{T}_1)$, $\mathfrak{s}_2 = X_2(\mathcal{T}_2)$ be infinite-dimensional non-finitary diagonal Lie algebras. Set $S_i = \mathrm{Stz}(\mathcal{S}_i)$, $S = \mathrm{GCD}(S_1, S_2)$, $R_i = \div(S_i, S)$, $\delta_i = \delta(\mathcal{T}_i)$, $C_i = \mathrm{Stz}(\mathcal{C}_i)$, $C = \mathrm{GCD}(C_1, C_2)$, $B_i = \div(C_i, C)$, and $\sigma_i = \sigma(\mathcal{T}_i)$ for $i = 1, 2$. Then $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$ if and only if the following conditions hold.

   1) R_1 is finite.
   2) $\mathfrak{s}_2$ is sparse if $\mathfrak{s}_1$ is sparse.

The statement of the theorem

3) If $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are non-sparse, both R_1 and R_2 are finite, and S is not divisible by an infinite power of any prime number, then $\epsilon\frac{R_1}{\delta_1} \leq \frac{R_2}{\delta_2}$ for ϵ as specified below. The inequality is strict if $\mathfrak{s}_1$ is pure and $\mathfrak{s}_2$ is dense. We have $\epsilon = 2$, except in the cases listed below, in which $\epsilon = 1$:
   - 3.1) $(X_1, X_2) = (C, C)$, (O, O), (C, A), (O, A), and $(X_1, X_2) = (A, A)$ with both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ being one-sided;
   - 3.2) $(X_1, X_2) = (A, A)$, B_1 is finite, either $\mathfrak{s}_1$ is one-sided and $\mathfrak{s}_2$ is two-sided non-symmetric or $\mathfrak{s}_2$ is two-sided weakly non-symmetric and $\mathfrak{s}_1$ is two-sided non-symmetric;
   - 3.3) $(X_1, X_2) = (A, A)$, B_1 is finite, both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two-sided strongly non-symmetric, either B_2 is infinite or C is divisible by an infinite power of any prime number;
   - 3.4) $(X_1, X_2) = (A, A)$, both B_1 and B_2 are finite, both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two-sided strongly non-symmetric, C is not divisible by an infinite power of any prime number, and $\frac{R_1\sigma_1}{B_1} \geq \frac{R_2\sigma_2}{B_2}$.

Ideas of the proof ($\mathrm{sl}(\infty) \to$ pure one-sided)

$$
\begin{array}{ccccccccc}
\mathrm{sl}(2) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(k) & \longrightarrow & \mathrm{sl}(k+1) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \theta_2} & & & & \downarrow{\scriptstyle \theta_k} & & \downarrow{\scriptstyle \theta_{k+1}} & & \\
\mathrm{sl}(n_1 n_2) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(n_1 \cdots n_k) & \longrightarrow & \mathrm{sl}(n_1 \cdots n_{k+1}) & \longrightarrow & \cdots
\end{array}
$$

We choose θ_k such that as $\mathrm{sl}(k)$-modules

$$V_k \downarrow \mathrm{sl}(k) \cong a_0^k \bigwedge^0(F_k) \oplus a_1^k \bigwedge^1(F_k) \oplus \cdots \oplus a_k^k \bigwedge^k(F_k).$$

$$
\begin{array}{c}
a_0^0 \\
a_0^1 \; a_1^1 \\
a_0^2 \; a_1^2 \; a_2^2 \\
\cdots
\end{array}
$$

with the conditions

$$a_i^k + a_{i+1}^k = n_k a_i^{k-1},\ k \geq 1 \text{ and } a_0^0 = 1.$$

Ideas of the proof (sparse one-sided $\not\to$ pure one-sided)

$$\begin{array}{ccccccc}
\mathrm{sl}(n_1) & \longrightarrow \cdots \longrightarrow & \mathrm{sl}(n_i) & \xrightarrow{(l_i,0,z_i)} & \mathrm{sl}(n_{i+1}) & \longrightarrow & \cdots \\
\downarrow \theta_1 & & \downarrow \theta_i & & \downarrow \theta_{i+1} & & \\
\mathrm{sl}(m_1) & \longrightarrow \cdots \longrightarrow & \mathrm{sl}(m_1 \cdots m_i) & \xrightarrow{(m_{i+1},0,0)} & \mathrm{sl}(m_1 \cdots m_{i+1}) & \longrightarrow & \cdots
\end{array}$$

$$V_i \downarrow \mathrm{sl}(n_i) \cong \bigoplus_{\lambda \in H_i} \underbrace{F_{n_i}^{\lambda} \oplus \cdots \oplus F_{n_i}^{\lambda}}_{t_\lambda}, \; d_i = \max_{\lambda \in H_i}(\lambda_1 - \lambda_{n_i}).$$

Using branching rules we prove that $d_i \geq d_{i+1}$. Denote $d = \lim d_i$.

Then $\lambda_{d+1} = \lambda_{d+2} = \cdots = \lambda_{n_i - d}$ for $\lambda \in H_i$ for large enough i, which yields $I(\theta_i) \leq \frac{c_0 n_i}{n_i^2 - 1} m_1 \cdots m_i$.

By calculating $I_{\mathrm{sl}(n_1)}^{\mathrm{sl}(m_1 \cdots m_i)}$ in two ways we get $\frac{l_1 \cdots l_{i-1}}{n_i} \geq c$.

Homomorphisms of diagonal Lie algebras

- Diagonal and non-diagonal homomorphisms
- Natural representations of diagonal Lie algebras

  A natural $\mathfrak{g}$-module is any non-zero $\mathfrak{g}$-module which can be constructed as a direct limit $V = \varinjlim V_n$, where V_n is the natural $\mathfrak{g}_n$-module.
- Inductive systems

The end

Thank you for your attention!