Gradings on Lie Algebras of Cartan and Melikian Type

Jason McGraw

Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John’s, Canada
jason.mcgraw@mun.ca

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Outline

1. Brief Overview of Gradings on Simple Lie Algebras
2. Brief Review of the Cartan Lie Algebras and Melikian Algebras
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Definition of a Grading

Definition

A grading $\Gamma$ on an algebra $A$ by a group $G$, also called a $G$-grading, is the decomposition of $A$ as the direct sum of subspaces $A_g$,

\[ \Gamma : A = \bigoplus_{g \in G} A_g, \]

such that $A_{g'}A_{g''} \subset A_{g'g''}$ for all $g', g'' \in G$. For $g \in G$, the subspace $A_g$ is called the homogeneous space of degree $g$, and any nonzero element $y \in A_g$ is called homogeneous of degree $g$. 
Group-Equivalent and Isomorphic Gradings

Definition

The set \( \text{Supp} \Gamma = \{ g \in G \mid A_g \neq 0 \} \) is called the support of the grading \( \Gamma : A = \bigoplus_{g \in G} A_g \). By \( \langle \text{Supp} \Gamma \rangle \) we denote the subgroup of \( G \) generated by \( \text{Supp} \Gamma \).

Note: We usually require that the grading group is generated by the support.

Definition

Two gradings \( A = \bigoplus_{g \in G} A_g \) and \( A = \bigoplus_{h \in G} A'_h \) by a group \( G \) on an algebra \( A \) are called group-equivalent if there exist \( \Psi \in \text{Aut} A \) and \( \theta \in \text{Aut} G \) such that \( \Psi(A_g) = A'_{\theta(g)} \) for all \( g \in G \). If \( \theta \) is the identity, we call the gradings isomorphic.

Our goal is to classify all gradings on an algebra up to group-equivalence or, if possible, up to isomorphism. The difference between isomorphic gradings is a matter of "relabelling" a basis of homogeneous elements.
Lemma

Let \( A = \bigoplus_{g \in G} A_g \) be a grading by a group \( G \) on an algebra \( A \) and \( \phi \) a homomorphism of \( G \). The associated \( \phi(G) \)-grading on \( A \) can be defined if one sets
\[
A = \bigoplus_{h \in \phi(G)} A_h,
\]
where
\[
A_h = \bigoplus_{g \in G, \phi(g) = h} A_g.
\]
□

Lemma

Let \( L \) be a simple Lie algebra. If \( L = \bigoplus_{g \in G} L_g \) is a \( G \)-grading of \( L \) such that the support generates \( G \), then \( G \) is abelian.
□

For the rest of the talk we assume that the grading group is finitely generated and abelian.
General Properties

Lemma

Let $A$ be an algebra over a field of characteristic $p$, $G$ a finitely generated abelian group (without $p$-torsion if $p > 0$) and $\Gamma : A = \bigoplus_{g \in G} A_g$ a $G$-grading on $A$. Then there is a homomorphism of $\hat{G}$ into $\text{Aut} A$ defined by the following action:

$$\chi \ast y = \chi(g)y, \quad \text{for all } y \in A_g, \quad g \in G, \quad \chi \in \hat{G}.$$

The image of $\hat{G}$ in $\text{Aut} A$ is a semisimple abelian algebraic subgroup (quasi-torus). Conversely, given a quasi-torus $Q$ in $\text{Aut} A$, we obtain the eigenspace decomposition of $A$ with respect to $Q$, which is a grading by the group of regular characters of the algebraic group $Q$, $G = \mathfrak{X}(Q)$.

For example, let $L = L_0 \oplus L_{\bar{1}}$ be a $\mathbb{Z}_2$-grading. Then $Q$ is the group of order 2 generated by $\psi \in \text{Aut} L$ such that $\psi(y) = y$ for all $y \in L_0$ and $\psi(y) = -y$ for all $y \in L_{\bar{1}}$. 
Lemma

Let $A$ be an algebra over a field of positive characteristic $p$, $G = \mathbb{Z}_p$ and $\Gamma : A = \bigoplus_{g \in G} A_g$ a $G$-grading on $A$. Then there is a homomorphism of $G = \langle \overline{1} \rangle$ into $\text{Der} A$ defined by the following. For all $\overline{i} \in \mathbb{Z}_p$ and $y \in A_{\overline{i}}$ set

$$\overline{1} \ast y = \overline{i}y.$$

Conversely, given a semisimple derivation with eigenvalues in $\mathbb{Z}_p \subset F$ we obtain the eigenspace decomposition of $A$ with respect to this derivation, which is a grading by $\mathbb{Z}_p$. 
General Properties

**Lemma**

Let $L$ be a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p$ and $G$ a group. If $\Gamma : L = \bigoplus_{g \in G} L_g$ is a $G$-grading and $\langle \text{Supp } \Gamma \rangle = G$ then the following are true:

- $G$ is a finitely generated abelian group;
- $G = G_1 \times G_2$ where $G_1$ is a group that has no $p$-torsion and $G_2$ is a $p$-group generated by $l$ elements where $l$ is minimal;
- there exists a quasi-torus $Q$ of $\text{Aut } A$ isomorphic to $\hat{G}_1$ and the subspaces $L'_{g_1} = \bigoplus_{g_2 \in G_2} L_{(g_1,g_2)}$ are the eigenspaces of $Q$;
- there exists an epimorphism $\phi : G_2 \to \mathbb{Z}_p^l$ and a set of $l$ semisimple derivations $\{D_i\}_{i=1}^l$ of $L$ such that the subspaces $L''_h = \bigoplus_{g_1 \in G_1, \phi(g_2)=h} L_{(g_1,g_2)}$, $h \in \mathbb{Z}_p^l$ are the eigenspaces with respect to $\{D_i\}_{i=1}^l$;
- $Q$ and the derivations $D_i$, $1 \leq i \leq l$, commute. 

$\square$
General Properties

Let $L$ be a finite dimensional simple Lie algebra $L$ and $\Gamma : L = \bigoplus_{g \in G} L_g$ be a grading by a group $G$ without $p$-torsion. Then there is a quasi-torus $Q$ of $\text{Aut} L$ isomorphic to $\hat{G}$ such that $L_g$ are the eigenspaces of $L$ with respect to $Q$. If $\Gamma' : L = \bigoplus_{g \in G} L'_g$ is a $G$-grading isomorphic to $\Gamma$ then by definition there is a $\Psi \in \text{Aut} L$ such that $L'_g = \Psi(L_g)$. The quasi-torus of $\text{Aut} L$ associated to the $\Gamma'$ grading is $\Psi Q \Psi^{-1}$.

It follows that if we can classify all quasi-tori of the automorphism group of a simple finite dimensional Lie algebra $L$, up to conjugation by an automorphism of $L$, then we classify all gradings by groups with no $p$-torsion up to isomorphism.
General Properties

The following theorem is a result of Platonov.

**Theorem**

*Any quasi-torus of an algebraic group is contained in the normalizer of a maximal torus.*

We will show that, up to conjugation, a quasi-tori of the automorphism group of a simple graded Cartan Lie algebra or Melikian algebra is contained in a maximal torus of automorphism group.
Most of the following lemmas and theorems can be found in *Simple Lie Algebras over Fields of Positive Characteristic, Volume I* by Helmut Strade.
Definition

Let $O(m; n)$ to be the commutative algebra

$$O(m; n) := \left\{ \sum_{0 \leq a \leq \tau(n)} \alpha(a)x^{(a)} \right\}$$

over a field of characteristic $p$, where $\tau(n) = (p^{n_1} - 1, \ldots, p^{n_m} - 1)$, with multiplication

$$x^{(a)}x^{(b)} = \binom{a + b}{a} x^{(a+b)} ,$$

where

$$\binom{a + b}{a} = \prod_{i=1}^{m} \binom{a_i + b_i}{a_i} = \prod_{i=1}^{m} \frac{(a_i + b_i)!}{a_i!b_i!}.$$
For convenience, let $x_i := x^{(\epsilon_i)}$, $\epsilon_i := (0, \ldots, 0, 1, 0 \ldots, 0)$ where the 1 is at the $i$-th position.

- The sequence of maps $\gamma_0$, $\gamma_1$, $\ldots$ on $O(m; n)_{(1)} := \left\{ \sum_{0 < a \leq \tau(n)} \alpha(a)x^{(a)} \right\}$ to $O(m; n)$, where $\gamma_r(x_i) = x_i^{(r)}$ for all $1 \leq i \leq m, r \geq 0$, defines a system of divided powers on $O(m; n)_{(1)}$.

- The set $\{x_i\}_{i=1}^m$ and the divided power maps generate $O(m; n)$.

- Let $\partial_i \in \text{Der } O(m; n)$ such that $\partial_i(x_i^{(r)}) = x_i^{(r-1)}$, $\partial_i(x_j) = 0$ for $0 \leq r$, $i \neq j$. The derivation $\partial_i$ is a special derivation of $O(m; n)$.
**Definition**

Let $\mathcal{W}(m; n)$ be the Lie algebra

$$\mathcal{W}(m; n) := \left\{ \sum_{1 \leq i \leq m} f_i \partial_i \mid f_i \in O(m; n) \right\}$$

with the commutator defined by

$$[f \partial_i, g \partial_j] = f(\partial_i g) \partial_j - g(\partial_j f) \partial_i, \quad f, g \in O(m; n).$$

The Lie algebras $\mathcal{W}(m; n)$ can be viewed as Lie algebras of special derivations of $O(m; n)$ endowed with the system of divided powers mentioned previously. These algebras are called the Witt algebras.
The remaining simple graded Cartan type Lie algebras are subalgebras of $W(m; n)$. When dealing with the Hamiltonian and contact algebras in $m$ variables (types $H(m; n)$ and $K(m; n)$ on the next slide), it is useful to introduce the following notation:

\[
i' = \begin{cases} 
i + r, & \text{if } 1 \leq i \leq r, \\
i - r, & \text{if } r + 1 \leq i \leq 2r;\end{cases}
\]

\[
\sigma(i) = \begin{cases} 
1, & \text{if } 1 \leq i \leq r, \\
-1, & \text{if } r + 1 \leq i \leq 2r;\end{cases}
\]

where $m = 2r$ in the case of $H(m; n)$ and $2r + 1$ in the case of $K(m; n)$. We will also need the following differential forms:

\[
\omega_S := dx_1 \wedge \cdots \wedge dx_m, \quad m \geq 3;
\]

\[
\omega_H := \sum_{i=1}^{r} dx_i \wedge dx_i', \quad m = 2r;
\]

\[
\omega_K := dx_m + \sum_{i=1}^{2r} \sigma(i)x_i dx_i', \quad m = 2r + 1.
\]
Simple Graded Cartan Lie Algebras

Definition

We define the special, Hamiltonian and contact algebras as follows:

\[
S(m; n) := \{D \in W(m; n) \mid D(\omega_S) = 0\}, \quad m \geq 3,
\]
\[
H(m; n) := \{D \in W(m; n) \mid D(\omega_H) = 0\}, \quad m = 2r,
\]
\[
K(m; n) := \{D \in W(m; n) \mid D(\omega_K) \in O(m; n)\omega_K\}, \quad m = 2r + 1,
\]
respectively.

The above algebras are not simple in general.

Lemma

The Lie algebras \(S(m; n)^{(1)}\), \(H(m; n)^{(2)}\) and \(K(m; n)^{(1)}\) are simple.
Melikian Algebras

The next type of simple Lie algebras we will consider are the Melikian algebras which are defined over fields of characteristic 5. Set

\[ \tilde{W}(2; \mathfrak{n}) = O(2; \mathfrak{n})\tilde{\partial}_1 + O(2; \mathfrak{n})\tilde{\partial}_2, \]

\[ f_1\tilde{\partial}_1 + f_2\tilde{\partial}_2 := f_1\tilde{\partial}_1 + f_2\tilde{\partial}_2 \text{ for all } f_1, f_2 \in O(2; \mathfrak{n}) \text{ and } \text{div} : W(m; \mathfrak{n}) \to O(m; \mathfrak{n}) \]

the linear map defined by

\[ \text{div}(x^{(a)}\partial_i) = \partial_i(x^{(a)}) = x^{(a-\varepsilon_i)}. \]
Melikian Algebras

Definition

Let \( M(2; n) := O(2; n) \oplus W(2; n) \oplus \tilde{W}(2; n) \) be the algebra over a field \( F \) of characteristic 5 whose multiplication is defined by the following equations. For all \( D, E \in W(2; n), f, f_i, g_i \in O(2; n) \) we set

\[
\begin{align*}
[D, \tilde{E}] &:= [D, E] + 2 \text{div}(D) \tilde{E}, \\
[D, f] &:= D(f) - 2 \text{div}(D)f, \\
[f, \tilde{E}] &:= fE \\
[f_1, f_2] &:= 2(f_1 \partial_1(f_2) - f_2 \partial_1(f_1))\tilde{\partial}_2 + 2(f_2 \partial_2(f_1) - f_1 \partial_2(f_2))\tilde{\partial}_1. \\
[f_1 \tilde{\partial}_1 + f_2 \tilde{\partial}_2, g_1 \tilde{\partial}_1 + g_2 \tilde{\partial}_2] &:= f_1 g_2 - f_2 g_1.
\end{align*}
\]

We refer to the Lie algebras \( M(2; n) \) as the \textit{Lie algebras of Melikian type}. 

Canonical Gradings on $O(m; n)$, $W(m; n)$

Definition

The $\mathbb{Z}$-gradings on $O(m; n)$ and $W(m; n)$,

$$O(m; n) = \bigoplus_{k \in \mathbb{Z}} O(m; n)_k,$$

$$W(m; n) = \bigoplus_{k \in \mathbb{Z}} W(m; n)_k,$$

where

$$O(m; n)_k = \text{Span}\{x^{(a)} | a_1 + \cdots + a_m = k\},$$

$$W(m; n)_k = \bigoplus_{i=1}^{m} O(m; n)_{k+1} \partial_i = \text{Span}\{x^{(a)} \partial_i | a_1 + \cdots + a_m = k+1, 1 \leq i \leq m\},$$

are called the canonical $\mathbb{Z}$-gradings on $O(m; n)$ and $W(m; n)$, respectively. We denote the canonical $\mathbb{Z}$-gradings on $O(m; n)$ and $W(m; n)$ by $\Gamma_O$ and $\Gamma_W$, respectively and their degrees by $\deg_O$ and $\deg_W$, respectively.
Canonical Grading on $M(2; n)$

**Definition**

Set

\[
\begin{align*}
\deg_M(x^{(a)} \partial_i) & := 3 \deg_W(x^{(a)} \partial_i), \\
\deg_M(x^{(a)} \tilde{\partial}_i) & := 3 \deg_W(x^{(a)} \partial_i) + 2, \\
\deg_M(x^{(a)}) & := 3 \deg_O(x^{(a)}) - 2.
\end{align*}
\]

The subspaces $M_k = \text{Span}\{y \in M(2; n) \mid \deg_M(y) = k\}$ for $k \in \mathbb{Z}$ define the *canonical $\mathbb{Z}$-grading on $M(2; n)$*.

Note: $W(2; n) = \bigoplus_{i \in \mathbb{Z}} M_{3i}$. Hence $W(2; n)$ is a graded subalgebra of $M(2; n)$ with respect to the canonical $\mathbb{Z}$-grading on $M(2; n)$. 
For example,

$$\deg_\mathcal{O}(x_1 x_2^{(2)}) = 3,$$

$$\deg_W(x_1 x_2^{(2)} \partial_2) = 3 - 1 = 2,$$

$$\deg_M(x_1 x_2^{(2)} \partial_2) = 3(2) = 6,$$

$$\deg_M(x_1 x_2^{(2)} \tilde{\partial}_2) = 3(2) + 2 = 8,$$

$$\deg_M(x_1 x_2^{(2)}) = 3(3) - 2 = 7,$$
Canonical Filtrations

Definition

For any \(\mathbb{Z}\)-grading \(\Gamma : A = \bigoplus_{k \in \mathbb{Z}} A_k\) on an algebra \(A\) with a finite support \(\text{Supp } \Gamma\) there is an induced filtration

\[ A_s = A_{(s)} \subset A_{(s-1)} \subset \cdots \subset A_{(q)} = A, \]

where \(A_{(k)} = \bigoplus_{l \geq k} A_l\) for \(k \in \mathbb{Z}\), \(q = \min(\text{Supp } \Gamma)\) and \(s = \max(\text{Supp } \Gamma)\). We call the induced filtrations of the canonical \(\mathbb{Z}\)-gradings on \(O(m; n), W(m; n)\) and \(M(2; n)\) the canonical filtrations.

For example,

\[
O(2; (1, 1))_{(2)} = \text{Span}\{x^{(a)} | a_1 + a_2 \geq 2\}, \\
W(2; (1, 1))_{(1)} = \text{Span}\{x^{(a)} \partial_1, x^{(a)} \partial_2 | a_1 + a_2 \geq 2\}.
\]
Continuous Automorphisms of $O(m; n)$

**Definition**

Let $\mathcal{A}(m, n)$ be the set of all $m$-tuples $(y_1, \ldots, y_m) \in O(m; n)^m$ where each

$$y_i = \sum_{0 < a \leq \tau(n)} \alpha_i(a)x^{(a)} \quad \text{and} \quad \alpha_i(p^l e_j) = 0, \quad \text{if} \quad n_i + l > n_j,$$

for which $\det(\alpha_j(e_i))_{1 \leq i, j \leq m} \in F^*$.

**Theorem**

*(Theorem 6.3.2 in SLAOFPC by Strade)*

Let $\text{Aut}_c O(m; n)$ be the set of maps $\rho$ from $O(m; n)$ to $O(m; n)$, defined by the tuples $(y_1, \ldots, y_m) \in \mathcal{A}(m, n)$ such that

$$\rho \left( \sum_{0 \leq a \leq \tau(n)} \alpha(a)x^{(a)} \right) = \sum_{0 \leq a \leq \tau(n)} \alpha(a) \prod_{i=1}^{m} (y_i)^{(a_i)}.$$

Then $\text{Aut}_c O(m; n)$ is a subgroup of $\text{Aut} O(m; n)$. 
Note: Each automorphism $\psi \in \text{Aut}_c(O(m; n))$ is a divided power automorphism. That is,

$$\psi(x_i^{(a_i)}) = (\psi(x_i))^{(a_i)}.$$ 

It follows that $\psi$ is defined by its action on $V := \text{Span}\{x_i \mid 1 \leq i \leq m\}$. Also, $\text{Aut}_c O(m; n)$ respects the canonical filtration on $O(m; n)$. That is $\psi(O(m; n)_{(k)}) = O(m; n)_{(k)}$. 


Autowmorphism of the Cartan Lie Algebras

Let $\Phi$ be the map from $\text{Aut}_c O(m; n)$ to $\text{Aut} W(m; n)$ defined on $\psi \in \text{Aut}_c O(m; n)$ by

$$\Phi(\psi)(D) = \psi \circ D \circ \psi^{-1},$$

where $D \in W(m; n)$ and the elements of $W(m; n)$ are viewed as derivations of $O(m; n)$.

**Theorem**

(Theorem 7.3.2 in SLAOFPC by Strade)

The map $\Phi : \text{Aut}_c O(m; n) \rightarrow W(m; n)$ is an isomorphism of groups provided that $(m; n) \neq (1; 1)$ if $p = 3$. Also, except for the case of $H(m, (n_1; n_2))^{(2)}$ with $m = 2$ and $\min\{n_1, n_2\} = 1$ if $p = 3$,

\[
\begin{align*}
\text{Aut } S(m; n)^{(1)} &= \Phi(\{\psi \in \text{Aut}_c O(m; n) \mid \psi(\omega_S) \in F^x \omega_S\}), \\
\text{Aut } H(m; n)^{(2)} &= \Phi(\{\psi \in \text{Aut}_c O(m; n) \mid \psi(\omega_H) \in F^x \omega_H\}), \\
\text{Aut } K(m; n)^{(1)} &= \Phi(\{\psi \in \text{Aut}_c O(m; n) \mid \psi(\omega_K) \in O(m; n)^x \omega_K\}).
\end{align*}
\]
Automorphisms

Automorphism of the Cartan Lie Algebras

Note: Aut $\mathcal{W}(m; n)$ respects the canonical filtration on $\mathcal{W}(m; n)$.

For convenience, set

\[
\mathcal{W}(m; n) := \Phi^{-1}(\text{Aut } \mathcal{W}(m; n)) = \text{Aut}_c O(m; n),
\]

\[
\mathcal{S}(m; n) := \Phi^{-1}(\text{Aut } \mathcal{S}(m; n)^{(1)}) = \{\psi \in \text{Aut}_c O(m; n) | \psi(\omega_S) \in F^\times \omega_S\},
\]

\[
\mathcal{H}(m; n) := \Phi^{-1}(\text{Aut } \mathcal{H}(m; n)^{(2)}) = \{\psi \in \text{Aut}_c O(m; n) | \psi(\omega_H) \in F^\times \omega_H\},
\]

\[
\mathcal{K}(m; n) := \Phi^{-1}(\text{Aut } \mathcal{K}(m; n)^{(1)}))
\]

\[
= \{\psi \in \text{Aut}_c O(m; n) | \psi(\omega_K) \in O(m; n)^\times \omega_K\}.
\]
Automorphisms of Melikian Algebras

Lemma

The following are true.

- Any automorphism $\Psi$ of $M(2; n)$ respects the canonical filtration of $M(2; n)$.
- For every automorphism $\psi$ of $W(2; n)$ there exists an automorphism $\psi_M$ of $M(2; n)$ which respects $W(2; n)$ and whose restriction to $W(2; n)$ is $\psi$.
- If $\Theta \in \text{Aut}_W M(2; n)$ is such that $\pi(\Theta) = \text{Id}_W$ then for $y \in M_k$, $k \in \mathbb{Z}$, there exists $\beta \in F^\times$ such that $\Theta(y) = \beta^i y$ and $\beta^3 = 1$.

Definition

Let $\text{Aut}_W M(2; n) = \{ \psi \in \text{Aut} M(2; n) ; \mid \psi(W(2; n)) = W(2; n) \}$ be the subgroup of automorphisms that leave $W(2; n)$ invariant and let $\pi : \text{Aut}_W M(2; n) \to \text{Aut} W(2; n)$ be the restriction map.
Tori of Aut $X^{(2)}(m; n)$

Lemma

The following groups are maximal tori of Aut $W(m; n)$, Aut $S(m; n)^{(1)}$, Aut $H(m; n)^{(2)}$ and Aut $K(m; n)^{(1)}$, respectively:

\[ T_W = T_S = \{ \Psi \in \text{Aut } W(m; n) \mid \Psi(x^{(a)} \partial_i) = t^a t_i^{-1} x^{(a)} \partial_i, \ t \in (F^\times)^m \}, \]

\[ T_H = \{ \Psi \in \text{Aut } W(m; n) \mid \Psi(x^{(a)} \partial_i) = t^a t_i^{-1} x^{(a)} \partial_i, \ t \in (F^\times)^m, \\
\quad t; t_i' = t_j t_j', \ 1 \leq i, j \leq r \}, \]

\[ T_K = \{ \Psi \in \text{Aut } W(m; n) \mid t^a t_i^{-1} x^{(a)} \partial_i, \ t \in (F^\times)^m, \\
\quad t; t_i' = t_j t_j' = t_m, \ 1 \leq i, j \leq r \}, \]

\[ T_M = \{ \Psi \in \text{Aut } M(2; n) \mid \Psi(x^{(a)} \partial_i) = t_1^{3a_1} t_2^{3a_2} t_i^{-3}, \\
\quad \Psi(x^{(a)} \tilde{\partial}_i) = t_1^{3a_1} t_2^{3a_2} t_i^{-3} t_1 t_2, \\
\quad \Psi(x^{(a)}) = t_1^{3a_1} t_2^{3a_2} t_1^{-1} t_2^{-1}, \\
\quad (t_1, t_2) \in (F^\times)^2 \}. \]
Corollary

Let $X = W, S, H$ or $K$. The maximal tori $T_X$ in $\text{Aut } X^{(2)}(m; n)$ described in previous lemma correspond, under the algebraic group isomorphism $\Phi$, to the following maximal tori in $X(m; n)$:

\[
T_W = T_S = \{ \psi \in \mathcal{W}(m; n) \mid \psi(x_i) = t_i x_i, \ t_i \in F^\times \},
\]

\[
T_H = \{ \psi \in \mathcal{W}(m; n) \mid \psi(x_i) = t_i x_i, \ t_i \in F^\times, \ t_i t_i' = t_j t_j' \},
\]

\[
T_K = \{ \psi \in \mathcal{W}(m; n) \mid \psi(x_i) = t_i x_i, \ t_i \in F^\times, \ t_i t_i' = t_j t_j' = t_m, \ 1 \leq i, j \leq r \}.
\]
Eigenspaces

Note: The eigenspace decomposition of $T_W$ is the direct sum of the subspaces $\text{Span}\{x^{(a)}\}$. Since $T_X \subset T_W$, the eigenspaces of $O(m; n)$ with respect to $T_X$ direct sums of $\text{Span}\{x^{(a)}\}$.

Similarly, The eigenspace decomposition of $T_W$ is the direct sum of the subspaces

$$\text{Span}\{x^{(a+\varepsilon i)}\partial_i \mid 1 \leq i \leq m\}$$

Since $T_X \subset T_W$, the eigenspaces of $W(m; n)$ with respect to $T_X$ (viewed as a subgroup of $T_W$) is direct sums of $\text{Span}\{x^{(a+\varepsilon i)}\partial_i \mid 1 \leq i \leq m\}$. 
**Definition**

We denote by \( \text{Aut}_0 O(m; n) \) the subgroup of \( \text{Aut}_c O(m; n) \) consisting of all \( \psi \) such that

\[
\psi(x_i) = \sum_{j=1}^{m} \alpha_{i,j} x_j, \quad \alpha_{i,j} \in F, \quad 1 \leq j \leq m.
\]

The group \( \text{Aut}_0 O(m; n) \) is canonically isomorphic to a subgroup of \( \text{GL}(V) = \text{GL}(m) \), which we denote by \( \text{GL}(m; n) \).

If \( n_i = n_j \) for \( 1 \leq i, j \leq m \) then \( \text{GL}(m; n) = \text{GL}(m) \), otherwise it is properly contained in \( \text{GL}(m) \). The condition for a tuple \( (y_1, \ldots, y_n) \) to be in \( \mathcal{A}(m, n) \),

\[
y_i = \sum_{0 < a} \alpha_i(a) x^{(a)} \quad \text{with} \quad \alpha_i(p^l \epsilon_j) = 0 \text{ if } n_i + l > n_j,
\]

imposes a *flag* structure on the vector space \( V = \text{Span}\{x_1, \ldots, x_m\} \).
Normalizers of $T_X$ for $X = W, S, H$

It is easy to see that for $1 \leq i \leq m$ the subspaces $\text{Span}\{x_i\}$ are eigenspaces of $T_W, T_S$ and $T_H$.

If $\psi \in N_{\mathcal{X}(m; n)}(T_X)$ then it sends an eigenspace of $T_X$ to an eigenspace. Since $\text{Aut}_c O(m; n)$ preserves the filtration and $O(m; n)_1 = V$, we have that if $\psi \in N_{\mathcal{X}(m; n)}(T_X)$ then we have $\psi(x_i) \in \text{Span}\{x_{ji}\}$ for some $1 \leq j_i \leq m$. Hence the normalizers of $T_X$ in $\mathcal{X}(m; n)$ are isomorphic to a subset of $m \times m$ monomial matrices in $\text{GL}(m; n) \cap \mathcal{X}(m; n)$. 
Normalizers of $T_X$ for $X = W, S, H$

**Lemma**

$\text{Aut}_0 O(m; n) \cap S(m; n) = \text{Aut}_0 O(m; n)$ and

$\text{Aut}_0 O(m; n) \cap \mathcal{H}(m; n) \cong (\text{Sp}(m)\{F^\times \text{Id}\}) \cap \text{GL}(m; n)$.

**Lemma**

Let $X = W, S$ or $H$. If $Q$ is a quasi-torus of $\mathcal{X}(m; n)$ then there is a $\psi \in \mathcal{X}(m; n)$ such that $\psi Q \psi^{-1} \subset T_X$.

**Corollary**

Let $X = W, S$ or $H$. If $Q$ is a quasi-torus of $\text{Aut} \, \mathcal{X}^{(2)}(m; n)$ then there is a $\psi \in \text{Aut} \, \mathcal{X}^{(2)}(m; n)$ such that $\psi Q \psi^{-1} \subset T_X$.
Normalizer of $T_K$

For $T_K$, the subspaces $\text{Span}\{x_i\}$ for $1 \leq i \leq 2r$ are eigenspaces and so is $\text{Span}\{x_m, x_i x_i' \mid 1 \leq i \leq 2r\}$. If $\psi \in N_{\mathcal{K}(m; n)}(T_K)$ then it sends an eigenspace of $\mathcal{K}(m; n)$ to an eigenspace. Since $\text{Aut}_c O(m; n)$ preserves the filtration, if $\psi \in N_{\mathcal{K}(m; n)}(T_K)$ then for $1 \leq i \leq 2r$ we have

$$\psi(\text{Span}\{x_i\}) = \text{Span}\{x_{j_i}\}$$

for some $1 \leq j_i \leq 2r$ or

$$\psi(\text{Span}\{x_m, x_i x_i' \mid 1 \leq i \leq r\}).$$

Since the dimension of $\text{Span}\{x_m, x_i x_i' \mid 1 \leq i \leq r\}$ is greater than one, $\psi(x_i) \in \text{Span}\{x_{j_i}\}$. Similarly,

$$\psi(x_m) \in \alpha x_m + \text{Span}\{x_i x_i' \mid 1 \leq i \leq r\}.$$
Recall that $\mathcal{H}(m; n) = \{ \psi \in \text{Aut}_c O(m; n) \mid \psi(\omega_H) \in F^\times \omega_H \}$ and $\mathcal{K}(m; n) = \{ \psi \in \text{Aut}_c O(m; n) \mid \psi(\omega_K) \in O(m; n)^\times \omega_K \}$. Noticing that $d(\omega_K) = 2\omega_H$ and that if $\psi \in N_{\mathcal{K}(m; n)}(T_K)$ then $\psi$ leaves $O(2r; (n_1, \ldots, n_{2r}))$ invariant we can prove the following lemma.

**Lemma**

*If $Q$ is a quasi-torus of $\mathcal{K}(m; n)$ then there is a $\psi \in \mathcal{K}(m; n)$ such that $\psi Q \psi^{-1} \subset T_K$.***

**Corollary**

*If $Q$ is a quasi-torus of $\text{Aut} K(m; n)^{(1)}$ then there is a $\psi \in \text{Aut} K(m; n)^{(1)}$ such that $\psi Q \psi^{-1} \subset T_K$.***
Normalizer of $T_M$

The eigenspaces of $T_M$ are

$$\text{Span}\{x^{(a)}\},$$
$$\text{Span}\{x^{(a+\varepsilon_i)}\partial_i \mid i = 1, 2\},$$
$$\text{Span}\{x^{(a+\varepsilon_i)}\widetilde{\partial}_i \mid i = 1, 2\}.$$  

Since the elements of an eigenspace of $T_M$ are homogeneous elements of the canonical $\mathbb{Z}$-grading on $M(2; n)$ and $\text{Aut } M(2; n)$ preserves the canonical filtration, the normalizer of $T_M$ preserves the $\mathbb{Z}$-grading. Since $W(2; n) = \bigoplus_{i \in \mathbb{Z}} M_{3i}$ and the restriction of $T_M$ to $W(2; n)$ is $T_W$, we have that $N_{\text{Aut } M(2; n)}(T_M) \subset \text{Aut}_W M(2; n)$ and the restriction of $N_{\text{Aut } M(2; n)}(T_M)$ on $W(2; n)$ is $N_{\text{Aut } W(2; n)}(T_W)$. Hence if we have a quasi-torus $Q$ in $N_{\text{Aut } M(2; n)}(T_M)$ we can conjugate $Q$ by an automorphism $\psi$ in $\text{Aut}_W M(2; n)$ such that the restriction of $\psi_M Q \psi_M^{-1}$ is in $T_W$.

Lemma

*If $Q$ is a quasi-torus of $\text{Aut } M(2; n)$ then there is a $\psi \in \text{Aut } M(2; n)$ such that $\psi Q \psi^{-1} \subset T_M$.  

*
Theorem

If $L = X^{(2)}(m; n)$ is a simple graded Cartan type Lie algebra or a Melikian algebra and $L = \bigoplus_{g \in G} L_g$ is a grading by a group $G$ with no $p$-torsion then the grading is isomorphic to the eigenspace decomposition with respect to quasi-torus $Q$ contained in $T_X$. 