Gradings on Lie Algebras of Cartan and Melikian Type

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Outline

Definition of a Grading

Definition

A *grading* Γ *on an algebra* A *by a group* G, also called a G-*grading*, is the decomposition of A as the direct sum of subspaces A_g,

$$\Gamma : A = \bigoplus_{g \in G} A_g,$$

such that $A_{g'} A_{g''} \subset A_{g' g''}$ for all $g', g'' \in G$. For $g \in G$, the subspace A_g is called the homogeneous space of degree g, and any nonzero element $y \in A_g$ is called *homogeneous of degree* g.

Group-Equivalent and Isomorphic Gradings

Definition

The set $\operatorname{Supp}\Gamma = \{g \in G \mid A_g \neq 0\}$ is called the *support* of the grading $\Gamma : A = \bigoplus_{g\in G} A_g$. By $\langle \operatorname{Supp}\Gamma \rangle$ we denote the subgroup of G generated by $\operatorname{Supp}\Gamma$.

Note: We usually require that the grading group is generated by the support.

Definition

Two gradings $A = \bigoplus_{g\in G} A_g$ and $A = \bigoplus_{h\in G} A'_h$ by a group G on an algebra A are called *group-equivalent* if there exist $\Psi \in \operatorname{Aut} A$ and $\theta \in \operatorname{Aut} G$ such that $\Psi(A_g) = A'_{\theta(g)}$ for all $g \in G$. If θ is the identity, we call the gradings *isomorphic*.

Our goal is to classify all gradings on an algebra up to group-equivalence or, if possible, up to isomorphism. The difference between isomorphic gradings is a matter of "relabelling" a basis of homogeneous elements.

General Properties

Lemma

Let $A = \bigoplus_{g \in G} A_g$ *be a grading by a group* G *on an algebra* A *and* ϕ *a homomorphism of* G. *The* associated $\phi(G)$-grading on A *can be defined if one sets* $A = \bigoplus_{h \in \phi(G)} \overline{A}_h$, *where* $\overline{A}_h = \bigoplus_{g \in G,\ \phi(g)=h} A_g$. □

Lemma

Let L *be a simple Lie algebra. If* $L = \bigoplus_{g \in G} L_g$ *is a* G*-grading of* L *such that the support generates* G, *then* G *is abelian.* □

For the rest of the talk we assume that the grading group is finitely generated and abelian.

General Properties

Lemma

Let A be an algebra over a field of characteristic p, G a finitely generated abelian group (without p-torsion if $p > 0$) and $\Gamma : A = \bigoplus_{g \in G} A_g$ a G-grading on A. Then there is a homomorphism of $\widehat{G}$ into $\operatorname{Aut} A$ defined by the following action:

$$\chi * y = \chi(g)y, \quad \text{for all } y \in A_g, \quad g \in G, \quad \chi \in \widehat{G}.$$

The image of $\widehat{G}$ in $\operatorname{Aut} A$ is a semisimple abelian algebraic subgroup (quasi-torus). Conversely, given a quasi-torus Q in $\operatorname{Aut} A$, we obtain the eigenspace decomposition of A with respect to Q, which is a grading by the group of regular characters of the algebraic group Q, $G = \mathfrak{X}(Q)$. □

For example, let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a $\mathbb{Z}_2$-grading. Then Q is the group of order 2 generated by $\psi \in \operatorname{Aut} L$ such that $\psi(y) = y$ for all $y \in L_{\bar{0}}$ and $\psi(y) = -y$ for all $y \in L_{\bar{1}}$.

General Properties

Lemma

Let A be an algebra over a field of positive characteristic p, $G = \mathbb{Z}_p$ and $\Gamma : A = \bigoplus_{g\in G} A_g$ a G-grading on A. Then there is a homomorphism of $G = \langle \bar{1} \rangle$ into $\operatorname{Der} A$ defined by the following. For all $\bar{i} \in \mathbb{Z}_p$ and $y \in A_{\bar{i}}$ set

$$\bar{1} * y = \bar{i} y.$$

Conversely, given a semisimple derivation with eigenvalues in $\mathbb{Z}_p \subset F$ we obtain the eigenspace decomposition of A with respect to this derivation, which is a grading by $\mathbb{Z}_p$.

General Properties

Lemma

Let L be a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic p and G a group. If $\Gamma : L = \bigoplus_{g\in G} L_g$ is a G-grading and $\langle \operatorname{Supp}\Gamma\rangle = G$ then the following are true:

- *G is a finitely generated abelian group;*
- *$G = G_1 \times G_2$ where G_1 is a group that has no p-torsion and G_2 is a p-group generated by l elements where l is minimal;*
- *there exists a quasi-torus Q of $\operatorname{Aut} A$ isomorphic to $\widehat{G_1}$ and the subspaces $L'_{g_1} = \bigoplus_{g_2\in G_2} L_{(g_1,g_2)}$ are the eigenspaces of Q;*
- *there exists an epimorphism $\phi : G_2 \to \mathbb{Z}_p^l$ and a set of l semisimple derivations $\{D_i\}_{i=1}^l$ of L such that the subspaces $L''_h = \bigoplus_{g_1\in G_1,\ \phi(g_2)=h} L_{(g_1,g_2)}$, $h \in \mathbb{Z}_p^l$ are the eigenspaces with respect to $\{D_i\}_{i=1}^l$;*
- *Q and the derivations D_i, $1 \le i \le l$, commute.* □

General Properties

Let L be a finite dimensional simple Lie algebra L and $\Gamma : L = \bigoplus_{g \in G} L_g$ be a grading by a group G without p-torsion. Then there is a quasi-torus Q of $\mathrm{Aut}\, L$ isomorphic to $\widehat{G}$ such that L_g are the eigenspaces of L with respect to Q.
If $\Gamma' : L = \bigoplus_{g \in G} L'_g$ is a G-grading isomorphic to Γ then by definition there is a $\Psi \in \mathrm{Aut}\, L$ such that $L'_g = \Psi(L_g)$. The quasi-torus of $\mathrm{Aut}\, L$ associated to the Γ' grading is $\Psi Q \Psi^{-1}$.

It follows that if we can classify all quasi-tori of the automorphism group of a simple finite dimensional Lie algebra L, up to conjugation by an automorphism of L, then we classify all gradings by groups with no p-torsion up to isomorphism.

General Properties

The following theorem is a result of Platonov.

Theorem

Any quasi-torus of an algebraic group is contained in the normalizer of a maximal torus.

We will show that, up to conjugation, a quasi-tori of the automorphism group of a simple graded Cartan Lie algebra or Melikian algebra is contained in a maximal torus of automorphism group.

Most of the following lemmas and theorems can be found in *Simple Lie Algebras over Fields of Positive Characteristic, Volume I* by Helmut Strade.

O(m;n)

Definition

Let $O(m;\underline{n})$ to be the commutative algebra

$$O(m;\underline{n}) := \left\{ \sum_{0 \le a \le \tau(\underline{n})} \alpha(a) x^{(a)} \right\}$$

over a field of characteristic p, where $\tau(\underline{n}) = (p^{n_1} - 1, \dots, p^{n_m} - 1)$, with multiplication

$$x^{(a)} x^{(b)} = \binom{a+b}{a} x^{(a+b)},$$

where $\binom{a+b}{a} = \prod_{i=1}^{m} \binom{a_i + b_i}{a_i} = \prod_{i=1}^{m} \frac{(a_i + b_i)!}{a_i! b_i!}$.

O(m;n)

For convenience, let $x_i := x^{(\epsilon_i)}$, $\epsilon_i := (0, \ldots, 0, 1, 0 \ldots, 0)$ where the 1 is at the i-th position.

- The sequence of maps $\gamma_0,\ \gamma_1, \ldots$ on $O(m;\underline{n})_{(1)} := \left\{ \sum_{0 < a \leq \tau(\underline{n})} \alpha(a) x^{(a)} \right\}$ to $O(m;\underline{n})$, where $\gamma_r(x_i) = x_i^{(r)}$ for all $1 \leq i \leq m$, $r \geq 0$, defines a system of divided powers on $O(m;\underline{n})_{(1)}$.
- The set $\{x_i\}_{i=1}^{m}$ and the divided power maps generate $O(m;\underline{n})$.
- Let $\partial_i \in \mathrm{Der}\, O(m;\underline{n})$ such that $\partial_i(x_i^{(r)}) = x_i^{(r-1)}$, $\partial_i(x_j) = 0$ for $0 \leq r$, $i \neq j$. The derivation ∂_i is a special derivation of $O(m;\underline{n})$.

W(m;n)

Definition

Let $W(m;\underline{n})$ be the Lie algebra

$$W(m;\underline{n}) := \left\{ \sum_{1 \le i \le m} f_i \partial_i \mid f_i \in O(m;\underline{n}) \right\}$$

with the commutator defined by

$$[f\partial_i, g\partial_j] = f(\partial_i g)\partial_j - g(\partial_j f)\partial_i, \quad f, g \in O(m;\underline{n}).$$

The Lie algebras $W(m;\underline{n})$ can be viewed as Lie algebras of special derivations of $O(m;\underline{n})$ endowed with the system of divided powers mentioned previously. These algebras are called the Witt algebras.

The remaining simple graded Cartan type Lie algebras are subalgebras of $W(m;\underline{n})$. When dealing with the Hamiltonian and contact algebras in m variables (types $H(m;\underline{n})$ and $K(m;\underline{n})$ on the next slide), it is useful to introduce the following notation:

$$i' = \begin{cases} i+r, & \text{if } 1 \le i \le r, \\ i-r, & \text{if } r+1 \le i \le 2r; \end{cases}$$

$$\sigma(i) = \begin{cases} 1, & \text{if } 1 \le i \le r, \\ -1, & \text{if } r+1 \le i \le 2r; \end{cases}$$

where $m = 2r$ in the case of $H(m;\underline{n})$ and $2r+1$ in the case of $K(m;\underline{n})$. We will also need the following differential forms:

$$\begin{aligned} \omega_S &:= dx_1 \wedge \cdots \wedge dx_m, && m \ge 3; \\ \omega_H &:= \sum_{i=1}^{r} dx_i \wedge dx_{i'}, && m = 2r; \\ \omega_K &:= dx_m + \sum_{i=1}^{2r} \sigma(i) x_i dx_{i'}, && m = 2r+1. \end{aligned}$$

Simple Graded Cartan Lie Algebras

Definition

We define the *special, Hamiltonian* and *contact algebras* as follows:

$$\begin{array}{lcll} S(m;\underline{n}) & := & \{D \in W(m;\underline{n}) \mid D(\omega_S) = 0\}, & m \geq 3, \\ H(m;\underline{n}) & := & \{D \in W(m;\underline{n}) \mid D(\omega_H) = 0\}, & m = 2r, \\ K(m;\underline{n}) & := & \{D \in W(m;\underline{n}) \mid D(\omega_K) \in O(m;\underline{n})\omega_K\}, & m = 2r+1, \end{array}$$

respectively.

The above algebras are not simple in general.

Lemma

The Lie algebras $S(m;\underline{n})^{(1)}$, $H(m;\underline{n})^{(2)}$ *and* $K(m;\underline{n})^{(1)}$ *are simple.*

Melikian Algebras

The next type of simple Lie algebras we will consider are the Melikian algebras which are defined over fields of characteristic 5. Set

$$\widetilde{W(2;\underline{n})} = \mathcal{O}(2;\underline{n})\widetilde{\partial_1} + \mathcal{O}(2;\underline{n})\widetilde{\partial_2},$$

$f_1\widetilde{\partial_1 + f_2}\partial_2 := f_1\widetilde{\partial_1} + f_2\widetilde{\partial_2}$ for all $f_1,\ f_2 \in \mathcal{O}(2;\underline{n})$ and $\mathrm{div} : W(m;\underline{n}) \to \mathcal{O}(m;\underline{n})$ the linear map defined by

$$\mathrm{div}(x^{(a)}\partial_i) = \partial_i(x^{(a)}) = x^{(a-\varepsilon_i)}.$$

Melikian Algebras

Definition

Let $M(2;\underline{n}) := O(2;\underline{n}) \oplus W(2;\underline{n}) \oplus \widetilde{W(2;\underline{n})}$ be the algebra over a field F of characteristic 5 whose multiplication is defined by the following equations. For all $D, E \in W(2;\underline{n})$, $f, f_i, g_i \in O(2;\underline{n})$ we set

$$
\begin{aligned}
&[D, \widetilde{E}] := \widetilde{[D, E]} + 2\,\mathrm{div}(D)\widetilde{E}, \\
&[D, f] := D(f) - 2\,\mathrm{div}(D) f, \\
&[f, \widetilde{E}] := fE \\
&[f_1, f_2] := 2(f_1\partial_1(f_2) - f_2\partial_1(f_1))\widetilde{\partial}_2 + 2(f_2\partial_2(f_1) - f_1\partial_2(f_2))\widetilde{\partial}_1. \\
&[f_1\widetilde{\partial}_1 + f_2\widetilde{\partial}_2, g_1\widetilde{\partial}_1 + g_2\widetilde{\partial}_2] := f_1 g_2 - f_2 g_1.
\end{aligned}
$$

We refer to the Lie algebras $M(2;\underline{n})$ as the *Lie algebras of Melikian type.*

Canonical Gradings on $O(m;\underline{n})$, $W(m;\underline{n})$

Definition

The $\mathbb{Z}$-gradings on $O(m;\underline{n})$ and $W(m;\underline{n})$,

$$O(m;\underline{n}) = \bigoplus_{k\in\mathbb{Z}} O(m;\underline{n})_k,$$

$$W(m;\underline{n}) = \bigoplus_{k\in\mathbb{Z}} W(m;\underline{n})_k,$$

where

$$O(m;\underline{n})_k = \mathrm{Span}\{x^{(a)} \mid a_1 + \cdots + a_m = k\},$$

$$W(m;\underline{n})_k = \bigoplus_{i=1}^{m} O(m;\underline{n})_{k+1}\partial_i = \mathrm{Span}\{x^{(a)}\partial_i \mid a_1+\cdots+a_m = k+1,\ 1\le i\le m\},$$

are called the *canonical* $\mathbb{Z}$*-gradings on* $O(m;\underline{n})$ and $W(m;\underline{n})$, respectively. We denote the canonical $\mathbb{Z}$-gradings on $O(m;\underline{n})$ and $W(m;\underline{n})$ by Γ_O and Γ_W, respectively and their degrees by $\deg_O$ and $\deg_W$, respectively.

Canonical Grading on $M(2;\underline{n})$

Definition

Set

$$\begin{aligned} \deg_M(x^{(a)}\partial_i) &:= 3\deg_W(x^{(a)}\partial_i), \\ \deg_M(x^{(a)}\widetilde{\partial}_i) &:= 3\deg_W(x^{(a)}\partial_i) + 2, \\ \deg_M(x^{(a)}) &:= 3\deg_O(x^{(a)}) - 2. \end{aligned}$$

The subspaces $M_k = \mathrm{Span}\{y \in M(2;\underline{n}) \mid \deg_M(y) = k\}$ for $k \in \mathbb{Z}$ define the *canonical $\mathbb{Z}$-grading on $M(2;\underline{n})$*.

Note: $W(2;\underline{n}) = \bigoplus_{i\in\mathbb{Z}} M_{3i}$. Hence $W(2;\underline{n})$ is a graded subalgebra of $M(2;\underline{n})$ with respect to the canonical $\mathbb{Z}$-grading on $M(2;\underline{n})$.

For example,

$$\deg_O(x_1 x_2^{(2)}) = 3,$$

$$\deg_W(x_1 x_2^{(2)} \partial_2) = 3 - 1 = 2,$$

$$\deg_M(x_1 x_2^{(2)} \partial_2) = 3(2) = 6,$$

$$\deg_M(x_1 x_2^{(2)} \widetilde{\partial}_2) = 3(2) + 2 = 8,$$

$$\deg_M(x_1 x_2^{(2)}) = 3(3) - 2 = 7,$$

Canonical Filtrations

Definition

For any $\mathbb{Z}$-grading $\Gamma : A = \bigoplus_{k\in\mathbb{Z}} A_k$ on an algebra A with a finite support $\operatorname{Supp}\Gamma$ there is an *induced filtration*

$$A_s = A_{(s)} \subset A_{(s-1)} \subset \cdots A_{(q)} = A,$$

where $A_{(k)} = \bigoplus_{l\geq k} A_l$ for $k \in \mathbb{Z}$, $q = \min(\operatorname{Supp}\Gamma)$ and $s = \max(\operatorname{Supp}\Gamma)$. We call the induced filtrations of the canonical $\mathbb{Z}$-gradings on $O(m;\underline{n})$, $W(m;\underline{n})$ and $M(2;\underline{n})$ the *canonical filtrations*.

For example,

$$\begin{aligned} O(2;(1,1))_{(2)} &= \operatorname{Span}\{x^{(a)} \mid a_1 + a_2 \geq 2\}, \\ W(2;(1,1))_{(1)} &= \operatorname{Span}\{x^{(a)}\partial_1, x^{(a)}\partial_2 \mid a_1 + a_2 \geq 2\}. \end{aligned}$$

Continuous Automorphisms of $O(m; n)$

Definition

Let $\mathfrak{A}(m, \underline{n})$ be the set of all m-tuples $(y_1, \ldots, y_m) \in O(m; \underline{n})^m$ where each

$$y_i = \sum_{0 < a \leq \tau(\underline{n})} \alpha_i(a) x^{(a)}, \quad \text{and } \alpha_i(p^l \epsilon_j) = 0, \text{ if } n_i + l > n_j,$$

for which $\det(\alpha_j(\epsilon_i))_{1 \leq i,j \leq m} \in F^*$.

Theorem

(Theorem 6.3.2 in SLAOFPC by Strade)
Let $\operatorname{Aut}_c O(m; \underline{n})$ *be the set of maps* ρ *from* $O(m; \underline{n})$ *to* $O(m; \underline{n})$*, defined by the tuples* $(y_1, \ldots, y_m) \in \mathfrak{A}(m, \underline{n})$ *such that*

$$\rho\left(\sum_{0 \leq a \leq \tau(\underline{n})} \alpha(a) x^{(a)}\right) = \sum_{0 \leq a \leq \tau(\underline{n})} \alpha(a) \prod_{i=1}^{m} (y_i)^{(a_i)}.$$

Then $\operatorname{Aut}_c O(m; \underline{n})$ *is a subgroup of* $\operatorname{Aut} O(m; \underline{n})$.

Divided Power Automorphisms

Note: Each automorphism $\psi \in \mathrm{Aut}_c(O(m;\underline{n}))$ is a divided power automorphism. That is,

$$\psi(x_i^{(a_i)}) = (\psi(x_i))^{(a_i)}.$$

It follows that ψ is defined by its action on $V := \mathrm{Span}\{x_i \mid 1 \leq i \leq m\}$. Also, $\mathrm{Aut}_c\, O(m;\underline{n})$ respects the canonical filtration on $O(m;\underline{n})$. That is $\psi(O(m;\underline{n})_{(k)}) = O(m;\underline{n})_{(k)}$.

Automorphism of the Cartan Lie Algebras

Let Φ be the map from $\mathrm{Aut}_c\, O(m;\underline{n})$ to $\mathrm{Aut}\, W(m;\underline{n})$ defined on $\psi \in \mathrm{Aut}_c\, O(m;\underline{n})$ by

$$\Phi(\psi)(D) = \psi \circ D \circ \psi^{-1},$$

where $D \in W(m;\underline{n})$ and the elements of $W(m;\underline{n})$ are viewed as derivations of $O(m;\underline{n})$.

Theorem

(Theorem 7.3.2 in SLAOFPC by Strade)
The map $\Phi : \mathrm{Aut}_c\, O(m;\underline{n}) \to W(m;\underline{n})$ *is an isomorphism of groups provided that* $(m;\underline{n}) \neq (1;1)$ *if* $p = 3$. *Also, except for the case of* $H(m,(n_1;n_2))^{(2)}$ *with* $m = 2$ *and* $\min\{n_1,n_2\} = 1$ *if* $p = 3$,

$$\begin{aligned}
\mathrm{Aut}\, S(m;\underline{n})^{(1)} &= \Phi(\{\psi \in \mathrm{Aut}_c\, O(m;\underline{n}) \mid \psi(\omega_S) \in F^{\times}\omega_S\}),\\
\mathrm{Aut}\, H(m;\underline{n})^{(2)} &= \Phi(\{\psi \in \mathrm{Aut}_c\, O(m;\underline{n}) \mid \psi(\omega_H) \in F^{\times}\omega_H\}),\\
\mathrm{Aut}\, K(m;\underline{n})^{(1)} &= \Phi(\{\psi \in \mathrm{Aut}_c\, O(m;\underline{n}) \mid \psi(\omega_K) \in O(m;\underline{n})^{\times}\omega_K\}).
\end{aligned}$$

□

Automorphism of the Cartan Lie Algebras

Note: Aut $W(m;\underline{n})$ respects the canonical filtration on $W(m;\underline{n})$.

For convenience, set

$$\begin{aligned}
\mathcal{W}(m;\underline{n}) &:= \Phi^{-1}(\operatorname{Aut} W(m;\underline{n})) = \operatorname{Aut}_c O(m;\underline{n}),\\
\mathcal{S}(m;\underline{n}) &:= \Phi^{-1}(\operatorname{Aut} S(m;\underline{n})^{(1)}) = \{\psi \in \operatorname{Aut}_c O(m;\underline{n}) \mid \psi(\omega_S) \in F^{\times}\omega_S\},\\
\mathcal{H}(m;\underline{n}) &:= \Phi^{-1}(\operatorname{Aut} H(m;\underline{n})^{(2)}) = \{\psi \in \operatorname{Aut}_c O(m;\underline{n}) \mid \psi(\omega_H) \in F^{\times}\omega_H\},\\
\mathcal{K}(m;\underline{n}) &:= \Phi^{-1}(\operatorname{Aut} K(m;\underline{n})^{(1)})\\
&= \{\psi \in \operatorname{Aut}_c O(m;\underline{n}) \mid \psi(\omega_K) \in O(m;\underline{n})^{\times}\omega_K\}.
\end{aligned}$$

Automorphisms of Melikian Algebras

Lemma

The following are true.

- *Any automorphism* Ψ *of* $M(2;\underline{n})$ *respects the canonical filtration of* $M(2;\underline{n})$.
- *For every automorphism* ψ *of* $W(2;\underline{n})$ *there exists an automorphism* ψ_M *of* $M(2;\underline{n})$ *which respects* $W(2;\underline{n})$ *and whose restriction to* $W(2;\underline{n})$ *is* ψ.
- *If* $\Theta \in \mathrm{Aut}_W M(2;\underline{n})$ *is such that* $\pi(\Theta) = \mathrm{Id}_W$ *then for* $y \in M_k$, $k \in \mathbb{Z}$, *there exists* $\beta \in F^\times$ *such that* $\Theta(y) = \beta^i y$ *and* $\beta^3 = 1$. □

Definition

Let $\mathrm{Aut}_W M(2;\underline{n}) = \{\Psi \in \mathrm{Aut}\, M(2;\underline{n});\, |\, \Psi(W(2;\underline{n})) = W(2;\underline{n})\}$ be the subgroup of automorphisms that leave $W(2;\underline{n})$ invariant and let $\pi : \mathrm{Aut}_W M(2;\underline{n}) \to \mathrm{Aut}\, W(2;\underline{n})$ be the restriction map.

Tori of Aut $X^{(2)}(m;\underline{n})$

Lemma

The following groups are maximal tori of Aut $W(m;\underline{n})$, Aut $S(m;\underline{n})^{(1)}$, Aut $H(m;\underline{n})^{(2)}$ *and* Aut $K(m;\underline{n})^{(1)}$, *respectively:*

$$T_W = T_S = \{\Psi \in \operatorname{Aut} W(m;\underline{n}) \mid \Psi(x^{(a)}\partial_i) = \underline{t}^a t_i^{-1} x^{(a)}\partial_i,\ \underline{t} \in (F^\times)^m\},$$

$$T_H = \{\Psi \in \operatorname{Aut} W(m;\underline{n}) \mid \Psi(x^{(a)}\partial_i) = \underline{t}^a t_i^{-1} x^{(a)}\partial_i,\ \underline{t} \in (F^\times)^m,\ t_i t_{i'} = t_j t_{j'},\ 1 \le i,j \le r\},$$

$$T_K = \{\Psi \in \operatorname{Aut} W(m;\underline{n}) \mid \underline{t}^a t_i^{-1} x^{(a)}\partial_i,\ \underline{t} \in (F^\times)^m,\ t_i t_{i'} = t_j t_{j'} = t_m,\ 1 \le i,j \le r\},$$

$$T_M = \{\Psi \in \operatorname{Aut} M(2;\underline{n}) \mid \Psi(x^{(a)}\partial_i) = t_1^{3a_1} t_2^{3a_2} t_i^{-3},\ \Psi(x^{(a)}\widetilde{\partial}_i) = t_1^{3a_1} t_2^{3a_2} t_i^{-3} t_1 t_2,\ \Psi(x^{(a)}) = t_1^{3a_1} t_2^{3a_2} t_1^{-1} t_2^{-1},\ (t_1,t_2) \in (F^\times)^2\}.$$

□

Tori of $\mathcal{X}(m;\underline{n})$

Corollary

Let $X = W$, S, H or K. The maximal tori T_X in $\operatorname{Aut} X^{(2)}(m;\underline{n})$ described in previous lemma correspond, under the algebraic group isomorphism Φ, to the following maximal tori in $\mathcal{X}(m;\underline{n})$:

$$\begin{aligned}
\mathcal{T}_W = \mathcal{T}_S &= \{\psi \in \mathcal{W}(m;\underline{n}) \mid \psi(x_i) = t_i x_i,\ t_i \in F^\times\},\\
\mathcal{T}_H &= \{\psi \in \mathcal{W}(m;\underline{n}) \mid \psi(x_i) = t_i x_i,\ t_i \in F^\times,\ t_i t_{i'} = t_j t_{j'},\},\\
\mathcal{T}_K &= \{\psi \in \mathcal{W}(m;\underline{n}) \mid \psi(x_i) = t_i x_i,\ t_i \in F^\times,\ t_i t_{i'} = t_j t_{j'} = t_m,\\
&\qquad 1 \le i,j \le r\}.
\end{aligned}$$

Eigenspaces

Note: The eigenspace decomposition of $\mathcal{T}_W$ is the direct sum of the subspaces $\mathrm{Span}\{x^{(a)}\}$. Since $\mathcal{T}_X \subset \mathcal{T}_W$, the eigenspaces of $O(m;\underline{n})$ with respect to $\mathcal{T}_X$ direct sums of $\mathrm{Span}\{x^{(a)}\}$.

Similarly, The eigenspace decomposition of T_W is the direct sum of the subspaces

$$\mathrm{Span}\{x^{(a+\varepsilon_i)}\partial_i \mid 1 \leq i \leq m\}$$

. Since $T_X \subset T_W$, the eigenspaces of $W(m;\underline{n})$ with respect to T_X (viewed as a subgroup of T_W) is direct sums of $\mathrm{Span}\{x^{(a+\varepsilon_i)}\partial_i \mid 1 \leq i \leq m\}$.

Linear Automorphisms and Flag Structure

Definition

We denote by $\mathrm{Aut}_0\, O(m;\underline{n})$ the subgroup of $\mathrm{Aut}_c\, O(m;\underline{n})$ consisting of all ψ such that $\psi(x_i) = \sum_{j=1}^{m} \alpha_{i,j} x_j$, $\alpha_{i,j} \in F$, $1 \le j \le m$. The group $\mathrm{Aut}_0\, O(m;\underline{n})$ is canonically isomorphic to a subgroup of $\mathrm{GL}(V) = \mathrm{GL}(m)$, which we denote by $\mathrm{GL}(m;\underline{n})$.

If $n_i = n_j$ for $1 \le i, j \le m$ then $\mathrm{GL}(m;\underline{n}) = \mathrm{GL}(m)$, otherwise it is properly contained in $\mathrm{GL}(m)$. The condition for a tuple $(y_1, \ldots, y_n)$ to be in $\mathfrak{A}(m,\underline{n})$,

$$y_i = \sum_{0<a} \alpha_i(a) x^{(a)} \quad \text{with } \alpha_i(p^l \epsilon_j) = 0 \text{ if } n_i + l > n_j, \tag{1}$$

imposes a *flag* structure on the vector space $V = \mathrm{Span}\{x_1, \ldots, x_m\}$.

Normalizers of T_X for $X = W, S, H$

It is easy to see that for $1 \leq i \leq m$ the subspaces $\mathrm{Span}\{x_i\}$ are eigenspaces of $\mathcal{T}_W$, $\mathcal{T}_S$ and $\mathcal{T}_H$.

If $\psi \in N_{\mathcal{X}(m;\underline{n})}(\mathcal{T}_X)$ then it sends an eigenspace of $\mathcal{T}_X$ to an eigenspace. Since $\mathrm{Aut}_c\, O(m;\underline{n})$ preserves the filtration and $O(m;\underline{n})_1 = V$, we have that if $\psi \in N_{\mathcal{X}(m;\underline{n})}(\mathcal{T}_X)$ then we have $\psi(x_i) \in \mathrm{Span}\{x_{j_i}\}$ for some $1 \leq j_i \leq m$. Hence the normalizers of $\mathcal{T}_X$ in $\mathcal{X}(m;\underline{n})$ are isomorphic to a subset of $m \times m$ monomial matrices in $\mathrm{GL}(m;\underline{n}) \cap \mathcal{X}(m;\underline{n})$.

Normalizers of T_X for $X = W, S, H$

Lemma

$\mathrm{Aut}_0\, O(m;\underline{n}) \cap \mathcal{S}(m;\underline{n}) = \mathrm{Aut}_0\, O(m;\underline{n})$ *and*
$\mathrm{Aut}_0\, O(m;\underline{n}) \cap \mathcal{H}(m;\underline{n}) \cong (\mathrm{Sp}(m)\{F^{\times}\,\mathrm{Id}\}) \cap \mathrm{GL}(m;\underline{n})$.

Lemma

Let $X = W$, S or H. If Q is a quasi-torus of $\mathcal{X}(m;\underline{n})$ then there is a $\psi \in \mathcal{X}(m;\underline{n})$ such that $\psi Q \psi^{-1} \subset \mathcal{T}_X$.

Corollary

Let $X = W$, S or H. If Q is a quasi-torus of $\mathrm{Aut}\, X^{(2)}(m;\underline{n})$ then there is a $\psi \in \mathrm{Aut}\, X^{(2)}(m;\underline{n})$ such that $\psi Q \psi^{-1} \subset T_X$.

Normalizer of T_K

For $\mathcal{T}_K$, the subspaces $\mathrm{Span}\{x_i\}$ for $1 \leq i \leq 2r$ are eigenspaces and so is $\mathrm{Span}\{x_m, x_i x_{i'} \mid 1 \leq i \leq 2r\}$. If $\psi \in N_{\mathcal{K}(m;\underline{n})}(\mathcal{T}_K)$ then it sends an eigenspace of $\mathcal{K}(m;\underline{n})$ to an eigenspace. Since $\mathrm{Aut}_c\, O(m;\underline{n})$ preserves the filtration, if $\psi \in N_{\mathcal{K}(m;\underline{n})}(\mathcal{T}_K)$ then for $1 \leq i \leq 2r$ we have

$$\psi(\mathrm{Span}\{x_i\}) = \mathrm{Span}\{x_{j_i}\}$$

for some $1 \leq j_i \leq 2r$ or

$$\psi(\mathrm{Span}\{x_m,\, x_i x_{i'} \mid 1 \leq i \leq r\}).$$

Since the dimension of $\mathrm{Span}\{x_m,\, x_i x_{i'} \mid 1 \leq i \leq r\}$ is greater than one, $\psi(x_i) \in \mathrm{Span}\{x_{j_i}\}$. Similarly,

$$\psi(x_m) \in \alpha x_m + \mathrm{Span}\{x_i x_{i'} \mid 1 \leq i \leq r\}.$$

Normalizer of T_K

Recall that $\mathcal{H}(m;\underline{n}) = \{\psi \in \operatorname{Aut}_c O(m;\underline{n}) \mid \psi(\omega_H) \in F^\times \omega_H\}$ and $\mathcal{K}(m;\underline{n}) = \{\psi \in \operatorname{Aut}_c O(m;\underline{n}) \mid \psi(\omega_K) \in O(m;\underline{n})^\times \omega_K\}$. Noticing that $d(\omega_K) = 2\omega_H$ and that if $\psi \in N_{\mathcal{K}(m;\underline{n})}(T_K)$ then ψ leaves $O(2r;(n_1,\dots,n_{2r}))$ invariant we can prove the following lemma.

Lemma

If Q is a quasi-torus of $\mathcal{K}(m;\underline{n})$ then there is a $\psi \in \mathcal{K}(m;\underline{n})$ such that $\psi Q \psi^{-1} \subset \mathcal{T}_K$.

Corollary

If Q is a quasi-torus of $\operatorname{Aut} K(m;\underline{n})^{(1)}$ then there is a $\psi \in \operatorname{Aut} K(m;\underline{n})^{(1)}$ such that $\psi Q \psi^{-1} \subset T_K$.

Normalizer of T_M

The eigenspaces of T_M are

$$\mathrm{Span}\{x^{(a)}\},$$
$$\mathrm{Span}\{x^{(a+\varepsilon_i)}\partial_i \mid i = 1,2\},$$
$$\mathrm{Span}\{x^{(a+\varepsilon_i)}\widetilde{\partial}_i \mid i = 1,2\}.$$

Since the elements of an eigenspace of T_M are homogeneous elements of the canonical $\mathbb{Z}$-grading on $M(2;\underline{n})$ and $\mathrm{Aut}\, M(2;\underline{n})$ preserves the canonical filtration, the normalizer of T_M preserves the $\mathbb{Z}$-grading. Since $W(2;\underline{n}) = \bigoplus_{i\in\mathbb{Z}} M_{3i}$ and the restriction of T_M to $W(2;\underline{n})$ is T_W, we have that $N_{\mathrm{Aut}\, M(2;\underline{n})}(T_M) \subset \mathrm{Aut}_W\, M(2;\underline{n})$ and the restriction of $N_{\mathrm{Aut}\, M(2;\underline{n})}(T_M)$ on $W(2;\underline{n})$ is $N_{\mathrm{Aut}\, W(2;\underline{n})}(T_W)$. Hence if we have a quasi-torus Q in $N_{\mathrm{Aut}\, M(2;\underline{n})}(T_M)$ we can conjugate Q by an automorphism ψ in $\mathrm{Aut}_W\, M(2;\underline{n})$ such that the restriction of $\psi_M Q \psi_M^{-1}$ is in T_W.

Lemma

If Q is a quasi-torus of $\mathrm{Aut}\, M(2;\underline{n})$ then there is a $\psi \in \mathrm{Aut}\, M(2;\underline{n})$ such that $\psi Q \psi^{-1} \subset T_M$.

Conclusion

Theorem

If $L = X^{(2)}(m; \underline{n})$ is a simple graded Cartan type Lie algebra or a Melikian algebra and $L = \bigoplus_{g \in G} L_g$ is a grading by a group G with no p-torsion then the grading is isomorphic to the eigenspace decomposition with respect to quasi-torus Q contained in T_X.