The Classification of C*-algebras

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C*-algebras

Definition (Abstract C*-algebras)

A *C*-algebra* is a Banach algebra A with an involution that satisfies the condition

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Theorem (Gelfand-Naimark-Segal)

Any (abstract) C-algebra is $*$-isomorphic to a norm closed self-adjoint subalgebra of $B(\mathscr{H})$ for a Hilbert space $\mathscr{H}$.*

Remark

In general, the choice of $\mathscr{H}$ is highly nonunique.

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4. $\mathrm{C}_0(X)$, the algebra of complex valued continuous functions over a locally compact topological space X which vanish at the infinity.
5. Let X be a compact Hausdorff space and let α be an action of $\mathbb{Z}$. Then $\mathrm{C}(X) \rtimes_\alpha \mathbb{Z}$ is the universal C*-algebra generated by $\mathrm{C}(X)$ and a unitary u such that

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6. Rotation C*-algebra: Let $\theta \in [0, 1]$. The rotation C*-algebra A_θ is the universal C*-algebra generated by unitaries u and v satisfying
$$uv = e^{2\pi i\theta} vu.$$

Simple C*-algebras

A C*-algebra A is simple if the only two-sided closed ideal is $\{0\}$ and A itself.

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Remark

An *unital* C*-algebra is simple if and only if it is algebraic simple.

Commutative C*-algebras

Theorem (Gelfand-Naimark)

Any commutative C-algebra is canonically* $*$*-isomorphic to* $C_0(X)$ *for a locally compact Hausdorff space* X.

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Let a be a normal operator in $B(\mathscr{H})$, i.e., $a^*a = aa^*$. Then,

$$\mathrm{C}^*(I, a) \cong \mathrm{C}(\sigma(a)),$$

where $\sigma(a) = \{\lambda \in \mathbb{C};\ (a - \lambda I) \text{ is not invertible}\}$ is the spectrum of a.

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Remark

Thus, C*-algebras are regarded as “non-commutative spaces”.

Inductive limits

Given

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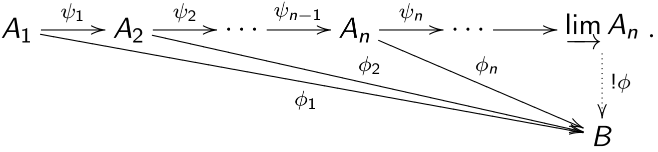
the *inductive limit* $\varinjlim (A_n, \psi_n)$ is the universal C*-algebra satisfying the following universal property:



Several naturally arising inductive limit C*-algebras

1. (Elliott-Evans) Let $\theta \in [0,1] \setminus \mathbb{Q}$. The rotation algebras A_θ (the universal C*-algebra generated by unitaries u and v satisfying $uv = e^{2\pi i\theta} vu$) is an inductive limit of $\bigoplus_i \mathrm{M}_{n_i}(\mathrm{C}(\mathbb{T}))$ (A$\mathbb{T}$-algebra).

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3. (Walters, Echterhoff-Lück-Phillips-Walters) For certain finite group actions on A_θ, the C*-algebra $A_\theta \rtimes_\alpha G$ is an inductive limit of finite dimensional C*-algebras (AF-algebras).

Several naturally arising inductive limit C*-algebras

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4. (Elliott-N) Certain extended rotation algebras (C*-algebras generated by a rotation algebra together with logarithms of the canonical unitaries) are AF.

Inductive limit vs. local approximation

Let $\mathcal{C}$ be a class of C*-algebras (e.g., finite dimensional C*-algebras, circle algebras, etc.). A C*-algebra A can be locally approximated by C*-algebras in $\mathcal{C}$ if for any finite subset $\mathcal{F} \subset A$, any $\varepsilon > 0$, there exists a sub-C*-algebra $C \subseteq A$ such that $C \in \mathcal{C}$ and $\mathcal{F} \subset_\varepsilon C$.

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- Inductive limit $\Rightarrow$ Local approximation.
- In general, Inductive limit $\not\Leftarrow$ Local approximation. (But true for certain C*-algebras, e.g., AF-algebras, A$\mathbb{T}$-algebras.)

K-groups of C*-algebras

Definition

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Consider $\mathrm{M}_\infty(A) = \bigcup_{n=1}^\infty \mathrm{M}_n(A)$, and denote by $\mathcal{D}(A)$ the equivalent classes of projections in $\mathrm{M}_\infty(A)$. $\mathcal{D}(A)$ is a semigroup with addition

$$[p] + [q] := [p \oplus q].$$

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- If $A = \varinjlim A_i$, then $\mathcal{D}(A) = \varinjlim \mathcal{D}(A_i)$.

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Example

1. Let $A = \mathrm{M}_n(\mathbb{C})$. Then $\mathcal{D}(A) = \{0, 1, 2, ...\}$.
2. Let $A = B(\mathscr{H})$ for a separable infinite dimensional Hilbert space $\mathscr{H}$. Then $\mathcal{D}(A) = \{0, 1, 2, ..., \infty\}$.

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The K_0-*group* of A (assuming to be unital, for convenience) is the Grothendieck enveloping group of $\mathcal{D}(A)$, i.e., the group of the formal differences of the elements of $\mathcal{D}(A)$.

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3. If $A = \bigotimes \mathrm{M}_2(\mathbb{C}) = \varinjlim \mathrm{M}_{2^n}(\mathbb{C})$, then $\mathrm{K}_0(A) = \mathbb{Z}[\frac{1}{2}] \subseteq \mathbb{Q}$ and $\mathcal{D}(A) = \mathbb{Z}^+[\frac{1}{2}] \subseteq \mathbb{Q}^+$.

Order-structure on K_0-groups

Let $\mathcal{D}(A)$ still denote the image of $\mathcal{D}(A)$ in $\mathrm{K}_0(A)$. For any stably finite unital C*-algebra A, the triple

$$(\mathrm{K}_0(A), \mathcal{D}(A), [1_A])$$

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is an order-unit group, i.e.,

1. $\mathcal{D}(A) - \mathcal{D}(A) = \mathrm{K}_0(A)$;
2. $\mathcal{D}(A) \cap (-\mathcal{D}(A)) = \{0\}$;
3. for any $\kappa \in \mathrm{K}_0(A)$, there exists n, such that $n[1_A] > \kappa$.

In this case, let us denote $\mathcal{D}(A)$ by $\mathrm{K}_0^+(A)$.

The classification of AF-algebras

Theorem (Elliott 1976)

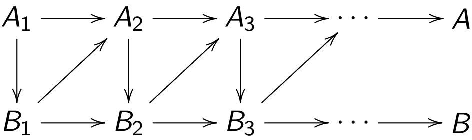
If A and B are AF-algebras and

$$\sigma : (K_0(A), K_0^+(A), [1_A]_0) \to (K_0(B), K_0^+(B), [1_B]_0)$$

is an isomorphism, then there is a $*$-isomorphism $\phi : A \to B$ such that $\phi_* = \sigma$.

The intertwining argument

Let $A = \varinjlim A_i$ and $B = \varinjlim B_i$, then $A \cong B$ if



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Theorem (Existence Theorem)

Let $A = \mathrm{M}_n(\mathbb{C})$ *and let* $B = \varinjlim B_i$ *be an AF-algebra. For any map*

$$\kappa : (K_0(A), K_0^+(A), [1]) \to (K_0(B), K_0^+(B), [1]),$$

there is a map $\phi : A \to B_i$ *such that* $[\phi]_0 = \kappa$.

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Theorem (Uniqueness Theorem)

Let A and B be finite dimensional C-algebras, and let ϕ and ψ be two maps from A to B. If $[A]_0 = [B]_0$, then, there is a unitary $u \in B$ such that*

$$\phi(a) = u^* \psi(a) u, \quad \forall a \in A.$$

A larger class of C*-algebras: A$\mathbb{T}$-algebras

Theorem (Elliott 1993)

The class of simple unital A$\mathbb{T}$-algebras of real rank zero *is classified by*

$$((K_0(A), K_0^+(A), [1]), K_1(A)).$$

Remark

A C*-algebra is of real rank zero if any self-adjoint element can be approximated be self-adjoint elements with finite spectrum.

K_1-group

Definition

An element $u \in A$ is a *unitary* if

$$uu^* = u^*u = 1.$$

Denote by $\mathrm{U}_n(A)$ the group of unitaries of $\mathrm{M}_n(A)$. Embed $\mathrm{U}_n(A)$ into $\mathrm{U}_{n+1}(A)$ by $u \mapsto \mathrm{diag}(u, 1)$, and consider the topological group

$$\mathrm{U}_\infty(A) := \bigcup_n \mathrm{U}_n(A).$$

Define

$$\mathrm{K}_1(A) = \mathrm{U}_\infty(A)/\mathrm{U}_\infty(A)_0,$$

where $\mathrm{U}_\infty(A)_0$ is the connected component of $\mathrm{U}_\infty(A)$ containing the unit.

AI-algebras

Theorem (Elliott 1993)

The class of simple unital AI-algebras (inductive limits of $\bigoplus_k M_{n_k}(C([0,1]))$) is classified by

$$((K_0(A), K_0^+(A), [1]), \mathrm{T}(A), r_A),$$

where $T(A)$ is the simplex of tracial states and r_A is the canonical pairing between $T(A)$ and $K_0(A)$.

A tracial state of A is a linear functional $\tau : A \to \mathbb{C}$ such that $\tau(aa^*) > 0$, $\tau(1) = 1$, and

$$\tau(ab) = \tau(ba), \quad \forall a, b \in A.$$

Any tracial state induces a positive linear map from $\mathrm{K}_0(A)$ to $\mathbb{R}$. This gives the canonical pairing between $\mathrm{T}(A)$ and $\mathrm{K}_0(A)$.

Elliott Invariant

Definition

The *Elliott invariant* of A is defined by

$$\mathrm{Ell}(A) := ((\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1_A]_0), \mathrm{K}_1(A), \mathrm{T}(A), r_A),$$

where $\mathrm{T}(A)$ is the simplex of tracial states and r_A is the canonical pairing between $\mathrm{T}(A)$ and $\mathrm{K}_0(A)$.

Classification of AH-algebras

Definition

A C*-algebra is an AH-algebra if it is an inductive limit of C*-algebras in form of

$$p\mathrm{M}_n(\mathrm{C}(X))p$$

with X a compact metrizable space, and p a projection in $\mathrm{M}_n(\mathrm{C}(X))$.

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Theorem (Elliott-Gong-Li)

Let A and B be two simple unital AH-algebras without dimension growth. If there is a map $\kappa : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$, then there exists a map $\phi : A \to B$ such that $[\phi]_* = \kappa$.

The range of Elliott invariant

An ordered group (G, G^*) is called a Riesz group if for any $a, b, c, d \in G$ with $a, b < c, d$, there exists $e \in G$ such that $a, b \leq e \leq c, d$.

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Theorem (Villadsen)

$((G, G^+, u), H, \Delta, r)$ is the invariant of an simple AH-algebra without dimension growth if and only if (G, G^*) is a simple weakly unperforated Riesz group with $G \neq \mathbb{Z}$, H is an abelian group, Δ is a Choquet simplex, and r preserves extreme points.

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Remark

If the condition on dimension growth is dropped, then there are some exotic examples of AH-algebras with perforation in K-group.

Axiomatic approach to AH-algebras (Lin)

A C*-algebra A is a *tracially AI-algebra* if for any nonzero positive element $a \in A$, any $\varepsilon > 0$, any finite subset $\{a_1, ..., a_n\} \subseteq A$,

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$$a_i \approx \begin{pmatrix} [a_i'] & \\ & \begin{bmatrix} & & \\ & pa_ip & \\ & & \end{bmatrix} \in I \end{pmatrix}$$

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Some applications

- (Lin-Phillips) Any simple higher dimensional noncommutative tori is a tracially AI-algebra (in fact a tracially AF-algebra), and hence is an A$\mathbb{T}$-algebra by the calculation of its K-groups.

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- (Lin-Phillips) Let X be an infinite compact metric space with finite covering dimension, and let h be a minimal homomorphism. Then the associated crosed product C*-algebra $A = \mathrm{C}(X) \rtimes_h \mathbb{Z}$ is an AH-algebra if the image of $\mathrm{K}_0(A)$ is dense in $\mathrm{Aff}(\mathrm{T}(A))$.

The axiomatic approach to the classification of C*-algebra has been generalized to certain inductive limit of subhomogeneous C*-algebras.

Theorem (N)

The class of simple separable nuclear tracially splitting interval algebras satisfying the UCT can be classified by the Elliott invariant.

Remark

The range of the Elliott Invariant of such C*-algebras is strictly larger than that of AH-algebras.

Some recent progresses

Definition

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Question

For certain class of C*-algebras, does $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ imply that $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$?

Denote by Q the UHF-algebra (AF-algebra) with $\mathrm{K}_0(Q) = \mathbb{Q}$.

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Theorem (W. Winter, Lin, Lin-N)

Let A and B be unital simple C-algebras such that $A \otimes Q$ and $B \otimes Q$ are AH-algebras without dimension growth. Then $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$.*

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Theorem

An AH-algebra A has no dimension growth if and only if $A \otimes \mathcal{Z} \cong A$.

Some applications

- The algebras $\{\mathrm{C}(M) \rtimes_\sigma \mathbb{Z}\}$, where M is a manifold and σ is a uniquely ergodic minimal diffeomorphism, are classifiable.

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- Simple unital inductive limits of locally trivial continuous field of matrix algebras (not necessary in the form of $\bigoplus_i p_i \mathrm{M}_{n_i}(\mathrm{C}(X_i))p_i$) are classified by the Elliott invariant.

AH-algebra with diagonal maps

A unital homomorphism

$$\varphi : \mathrm{C}(X) \to \mathrm{M}_n(\mathrm{C}(Y))$$

is called a diagonal map if there are continuous maps

$$\lambda_1, ..., \lambda_n : Y \to X$$

such that

$$\varphi(f) = \begin{pmatrix} f \circ \lambda_1 & & \\ & \ddots & \\ & & f \circ \lambda_n \end{pmatrix}, \quad \forall f \in \mathrm{C}(X).$$

Theorem (Elliott-Ho-Toms)

Let A be a simple unital AH-algebra with diagonal maps (without assumption on dimension growth). Then A has topological stable rank one, i.e., the invertible elements are dense.

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Theorem (N)

Let A be a simple unital AH-algebra with diagonal maps. If A has at most countably many extremal tracial states or projection separates traces, then A is an AH-algebra without dimension growth.