The Green ring of the generalized Taft algebra $H_{n,d}$

Yinhuo Zhang
(joint work with Libin Li)

University of Hasselt

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The Green ring of a Hopf algebra

Definition

Let H be a Hopf algebra. The representation ring $r(H)$ can be defined as follows. As an abelian group $r(H)$ is generated by the isomorphism classes $[V]$ of finite dimensional H-modules V modulo the relations $[M \oplus V] = [M] + [V]$. The multiplication of $r(H)$ is given by the tensor product of H-modules, that is, $[M][V] = [M \otimes V]$.

$r(H)$ is an associative ring. Note that $r(H)$ is a free abelian group with a $\mathbb{Z}$-basis $\{[V] | V \in \mathrm{ind}(H)\}$, where $\mathrm{ind}(H)$ stands for the set of of all finite dimensional indecomposable H-modules. Denote by $R(H)$ the associative k-algebra $k \otimes_{\mathbb{Z}} r(H)$.

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Outline

1. Indecomposable representations of $H_{n,d}$.
2. Generators and generating relations of the Green ring $r(H_{n,d})$.
3. Nilpotent elements of $r(H_{n,d})$.
4. The projective class ring of $H_{n,d}$.

Outline

The generalized Taft algebra $H_{n,d}$

Fix two integers $n, d \geqslant 2$, $q^n = 1$ and $o(q) = d$. Radford considered the generalized Taft Hopf algebra $H_{n,d}$ generated by two elements g and h subject to the relations:

$$g^n = 1,\ h^d = 0,\ hg = qgh.$$

The algebra $H_{n,d}$ is a Hopf algebra with comultiplication Δ, counit ε, and antipode S given by

$$\begin{array}{lll} \Delta(g) = g \otimes g, & \Delta(h) = 1 \otimes h + h \otimes g, & \varepsilon(g) = 1, \\ \varepsilon(h) = 0, & S(g) = g^{-1} = g^{n-1}, & S(h) = -q^{-1} g^{n-1} h. \end{array}$$

Note that $\dim H_{n,d} = dn$, and the set $\{g^i h^j | 0 \leqslant i \leq n-1, 0 \leq j \leqslant d-1\}$ forms a PBW basis for $H_{n,d}$.
In case q is a primitive n-th root of unity (i.e, $d = n$), then $H_n = H_{n,n}$ is the n^2-dimensional Taft (Hopf) algebra.

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Indecomposable $H_{n,d}$-modules

For $1 \leqslant l \leqslant d$ and $i \in \mathbb{Z}_n$, denote $M(l,i)$ by the k-vector space with a K-basis $\{v_0, v_1, \cdots, v_{l-1}\}$. Let ω be a primitive n-th root of unity so that $\omega^m = q$. Define an action of $H_{n,d}$ on $M(l,i)$ as follows:

$g \cdot v_j = \omega^i q^{-j} v_j$ for all $0 \leqslant j \leqslant l-1$ and

$$h \cdot v_j = \begin{cases} v_{j+1}, & 0 \leqslant j \leqslant l-2, \\ 0, & j = l-1. \end{cases}$$

Lemma (Cibils, 93)

The set $\{M(l,i) \mid i \in \mathbb{Z}_n, 1 \leqslant l \leqslant d\}$ forms a complete list of non-isomorphic indecomposable $H_{n,d}$-modules. Moreover, $M(l,i)$ is irreducible if and only if $l = 1$; and $M(l,i)$ is projective indecomposable if and only if $l = d$.

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Decomposition formulas

Lemma

We have the following isomorphisms:

$$M(1,i)\otimes M(l,r)\cong M(l,r)\otimes M(1,i)\cong M(l,r+i),$$
$$M(l,r)\cong M(1,r)\otimes M(l,0)\cong M(l,0)\otimes M(1,r)$$

for all $1\leqslant l\leqslant d$ *and* $r\in\mathbb{Z}_n$.

Lemma

We have the following isomorphisms:

1. $M(2,0)\otimes M(l,0)\cong M(l+1,0)\bigoplus M(l-1,-m)$ *for all* $2\leqslant l\leqslant d-1$ *and* $d>2$.
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Outline

1. Indecomposable representations of $H_{n,d}$
2. The Green ring of $H_{n,d}$
3. Nilpotent elements of $r(H_{n,d})$
4. **The projective class ring of $H_{n,d}$**

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Generators

Lemma

The following equations hold in $r(H_{n,d})$.

1. $[M(1,-1)]^n = 1$ *and* $[M(l,r)] = [M(1,-1)]^{n-r}[M(l,0)]$ *for all* $1 \leqslant l \leqslant d$ *and* $r \in \mathbb{Z}_n$.
2. $[M(l+1,0)] = [M(2,0)][M(l,0)] - [M(1,-1)]^m[M(l-1,0)]$ *for all* $2 \leqslant l \leqslant d-1$ *and* $d > 2$.
3. $[M(2,0)][M(d,0)] = (1+[M(1,-1)]^m)[M(d,0)]$.
4. $[M(d,0)][M(d,0)] = \sum_{i=0}^{d-1}[M(d,-im)]$.

Corollary

The Green ring $r(H_{n,d})$ is a commutative ring generated by $[M(1,-1)]$ and $[M(2,0)]$.

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The generalized Fibonacci polynomials

Let $F_s(y,z)$ be the generalized Fibonacci polynomials defined by

$$F_{s+2}(y,z) = zF_{s+1}(y,z) - yF_s(y,z)$$

for $s \geq 2$, while $F_0(y,z) = 0,\ F_1(y,z) = 1,\ F_2(y,z) = z$. These generalized Fibonacci polynomials were found appearing in the generating relations of the Green rings of the Taft algebras H_n. The general form of the polynomials is as follows:

Lemma

For $s \geq 2$ we have

$$F_s(y,z) = \sum_{i=0}^{\lceil (s-1)/2 \rceil} (-1)^i \begin{bmatrix} s-1-i \\ i \end{bmatrix} y^i z^{s-1-2i}.$$

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Generating relations

Theorem

The Green ring $r(H_{n,d})$ of $H_{n,d}$ is isomorphic to the quotient ring of the polynomial ring $\mathbb{Z}[y, z]$ modulo the ideal I generated by the following elements

$$y^n - 1,\ (z - y^m - 1)F_d(y^m, z).$$

The isomorphism is given by

$$\Phi : \mathbb{Z}[y, z]/I \to r(H_{n,d}),\ \Phi(y) = [M(1, -1)],\ \Phi(z) = [M(2, 0)]$$

Moreover, under the isomorphism, the polynomial $F_s(y^m, z)$ corresponds to $[M(s, 0)]$ for $2 \leq s \leq d$.

Relation to the Green ring of the Taft algebra

Theorem

As a $\mathbb{Z}$-algebra, the Green ring $r(H_{n,d})$ is isomorphic to the group algebra $r(H_d)[C_m]$ over $r(H_d)$, that is, $r(H_{n,d}) \cong r(H_d)[x]/(x^m - 1)$.

Precisely, we have

$$r(H_{n,d}) = r(H_d) \bigoplus r(H_d)[M(1,-1)] \bigoplus \cdots \bigoplus r(H_d)[M(1,-1)]^{(m-1)}.$$

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Outline

The roots of the generalized Fibonacci polynomial

Proposition

Let $s \geq 2$. Then the generalized Fibonacci polynomial $F_s(a, x)$ has $s-1$ distinct complex roots given by

$$x_j = 2\sqrt{a}\cos\frac{j\pi}{s}, \quad 1 \leq j \leq s-1.$$

Moreover, for every $1 \leq j \leq s-1$, there exists a $2s$-th root $\eta_j \neq 1$ of unity, such that

$$x_j = 2\sqrt{a}\cos\frac{\pi j}{s} = \sqrt{a}(\eta_j + \eta_j^{-1}).$$

Lemma

The following system of equations

$$\begin{cases} y^n = 1, \\ (z - y^m - 1)F_d(y^m, z) = 0 \end{cases}$$

has $dn - n + m$ distinct solutions in $\mathbb{C}$, where $m = n/d$, and the solutions are given by

$$\begin{aligned} \mathfrak{T} = \ & \{(\omega_k, 2) \,|\, 0 \le k \le n-1, d|k\} \\ & \cup\{(\omega_k, \sigma_{k,j}) \,|\, 0 \le k \le n-1, 1 \le j \le d-1\}. \end{aligned} \tag{1}$$

where $\omega_k = cos\frac{2k\pi}{n} + i\ sin\frac{2k\pi}{n}$, and $\sigma_{k,j} = 2\sqrt{\omega_k^m}\ cos\frac{\pi j}{d},\ \ 1 \le j \le d-1$ are the distinct roots of $F_d(\omega_k^m, x)$.

Nilpotent elements

Proposition

For $d \geq 2$, the Green algebra $R(H_{n,d})$ has exactly $nd - n + m$ irreducible modules given by $\mathbb{C}_{\lambda,\mu}$, where $\mathbb{C}_{\lambda,\mu} = \mathbb{C}$ as vector spaces, and the action is given by $y \cdot 1 = \lambda 1, z \cdot 1 = \mu 1$, for $(\lambda, \mu) \in \mathfrak{T}$.

Theorem

Let $d \geq 2$. The set of nilpotent elements in $r(H_{n,d})$ is equal to

$$\langle [M(d,i)] - [M(d,j)] \mid i \equiv j (mod \ \ m) \rangle,$$

that is, the Jacobson radical $J(r(H_{n,d}))$ of $r(H_{n,d})$ has a $\mathbb{Z}$-basis

$$\{[M(d, im+j)] - [M(d, (i-1)m+j)] \mid 1 \leq i \leq d-1, 0 \leq j \leq m-1\}.$$

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Proof.

- For $i \equiv j (mod\ m)$, we have that $([M(d,i)] - [M(d,j)])^2 = 0$.
- The set $\{[M(d, im+j)] - [M(d,(i-1)m+j)] \mid 1 \leq i \leq d-1, 0 \leq j \leq m-1\}$ is independent over $\mathbb{Z}$.
- $dim_{\mathbb{C}}(J(R(H_{n,d}))) = n - m$ due to the above proposition
- The set $\{[M(d, im+j)] - [M(d,(i-1)m+j)] \mid 1 \leq i \leq d-1, 0 \leq j \leq m-1\}$ forms a $\mathbb{Z}$-basis of $J(r(H_{n,d}))$.

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Corollary

The Jacobson radical $J(r(H_{n,d}))$ is a principal ideal generated by the element $([M(1,m)]-1)[M(d,0)]$.

Proof.

For $1 \leq i \leq d-1, 0 \leq j \leq m-1$, by Lemma 2.2, we have

$$\begin{aligned} &[M(d, im+j)] - [M(d,(i-1)m+j)] \\ =&([M(1, im+j)] - [M(1,(i-1)m+j)])[M(d,0)] \\ =&[M(1,(i-1)m+j)]([M(1,m) - 1])[M(d,0)]. \end{aligned}$$

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Indecomposable modules over $R(H_{n,d})$

For each k, $0 \leq k \leq n-1$, such that $d \nmid k$, write ω_k for $cos\frac{2k\pi}{n} + isin\frac{2k\pi}{n}$, an n-th root of unity, and let $V(k)$ be a 2-dimensional $\mathbb{C}$-vector space with a basis $\{v_1, v_2\}$. Define an action of the Green algebra $R(H_{n,d})$ on $V(k)$ as follows:

$$y \cdot v_i = \omega_k v_i,\ z \cdot v_1 = (1+\omega_k^m)v_1,\ z \cdot v_2 = v_1 + (1+\omega_k^m)v_2.$$

Theorem

Let V be a finite dimensional indecomposable and reducible module of the Green algebra $R(H_{n,d})$. Then there exists k, $0 \leq k \leq n-1$, and $d \nmid k$, such that $V \cong V(k)$. Thus The set $\{\mathbb{C}_{\lambda,\mu}, V(k) \,|(\lambda,\mu) \in \mathfrak{T}, 0 \leq k \leq n-1, d \nmid k\}$ forms a complete list of finite dimensional indecomposable representations of $R(H_{n,d})$ with cardinal number nd.

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Outline

The projective class ring of $H_{n,d}$

- The projective class ring $p(H_{n,d})$ of $H_{n,d}$ is the subalgebra of $r(H_{n,d})$ generated by the projective and semisimple representations of $H_{n,d}$.
- $p(H_{n,d})$ has a $\mathbb{Z}$-basis $\{[M(1,i)],[M(d,i)]\,|\,0 \le i \le n-1\}$.
- Cibils determined the structure of $P(H_d)$ of the Taft algebra H_d, which is isomorphic to $\mathbb{C}^2 \times \mathbb{C}[\epsilon]^{d-1}$, where $\mathbb{C}[\epsilon]$ is the algebra of dual numbers $\mathbb{C}[x]/(x^2)$.
- Assume $d \ge 2$. Then as a $\mathbb{Z}$-algebra, $p(H_{n,d})$ is generated by $[M(1,-1)]$ and $[M(d,0)]$, and is isomorphic to $\mathbb{Z}[y,z]/I$ with I generated by the following elements:

$$y^n - 1,\ \ z^2 - (1 + y^m + y^{2m} + \cdots + y^{(d-1)m})z.$$

- $r(H_{n,d})$ and $p(H_{n,d})$ have the same set of nilpotent elements.

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- $r(H_{n,d})$ and $p(H_{n,d})$ have the same set of nilpotent elements.

The stable Green ring of $H_{n,d}$

- The stable Green ring was introduced in the study of Green rings for the modular representation theory of finite groups. It is a quotient of the Green ring modulo all projective representations.
- The stable Green ring $St(H_{n,d})$ of $H_{n,d}$ is generated by $[M(1,-1)]$ and $[M(2,0)]$ and is isomorphic to $\mathbb{Z}[y,z]/J$, where the ideal J is generated by the following elements:

$$y^n - 1, \ F_d(y^m, z).$$

- $St(H_{n,d})$ is semiprimitive.

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Thank you for your attention!