COMPATIBLE ELEMENTS FOR A TRIDIAGONAL PAIR

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Before we can begin to state results, we need to briefly review the following notions:

- The Lie algebra $\mathfrak{sl}_2$
- The $\mathfrak{sl}_2$ loop algebra
- Tridiagonal pairs

THE LIE ALGEBRA $\mathfrak{sl}_2$

Let $\mathfrak{sl}_2$ denote the Lie algebra over $\mathbb{F}$ with basis e, f, h and Lie bracket

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f.$$

THE $\mathfrak{sl}_2$ LOOP ALGEBRA

Let t denote an indeterminate. Let $\mathbb{F}[t, t^{-1}]$ denote the $\mathbb{F}$-algebra consisting of all Laurent polynomials in t that have coefficients in $\mathbb{F}$. Let $L(\mathfrak{sl}_2)$ denote the Lie algebra over $\mathbb{F}$ consisting of the $\mathbb{F}$-vector space $\mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}]$ and Lie bracket

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab, \quad u, v \in \mathfrak{sl}_2, \quad a, b \in \mathbb{F}[t, t^{-1}].$$

We call $L(\mathfrak{sl}_2)$ the $\mathfrak{sl}_2$ *loop algebra*.

TRIDIAGONAL PAIRS

Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *tridiagonal pair* on V we mean an ordered pair (A, B), where $A, B \in \mathrm{End}(V)$ satisfy the following four conditions:

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1. each of A, B is diagonalizable;
2. there exists an ordering $\{V_i\}_{i=0}^{d}$ of the eigenspaces of A such that $BV_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$, where $V_{-1} = 0$ and $V_{d+1} = 0$;

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3. there exists an ordering $\{V_i'\}_{i=0}^{\delta}$ of the eigenspaces of B such that $AV_i' \subseteq V_{i-1}' + V_i' + V_{i+1}'$ for $0 \leq i \leq \delta$, where $V_{-1}' = 0$ and $V_{\delta+1}' = 0$;

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3. there exists an ordering $\{V'_i\}_{i=0}^{\delta}$ of the eigenspaces of B such that $AV'_i \subseteq V'_{i-1} + V'_i + V'_{i+1}$ for $0 \leq i \leq \delta$, where $V'_{-1} = 0$ and $V'_{\delta+1} = 0$;
4. there is no subspace W of V such that $AW \subseteq W$, $BW \subseteq W$, $W \neq 0$, $W \neq V$.

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Example

Let $\mathfrak{a}$ and $\mathfrak{h}$ be two semisimple elements that generate $\mathfrak{sl}_2$, and let V be a finite-dimensional irreducible $\mathfrak{sl}_2$-module. Then $\mathfrak{a}, \mathfrak{h}$ act as a tridiagonal pair on V.

A REMARK ABOUT $\mathfrak{sl}_2$

$\mathfrak{sl}_2$ is isomorphic to the Lie algebra over $\mathbb{F}$ that has generators $\mathfrak{a}, \mathfrak{h}$ and relations

$$[\mathfrak{a}, [\mathfrak{a}, \mathfrak{h}]] = 4\mathfrak{h}, \qquad [\mathfrak{h}, [\mathfrak{h}, \mathfrak{a}]] = 4\mathfrak{a}.$$

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The inverse of this isomorphism is given by

$$e \mapsto \frac{[\mathfrak{h}, \mathfrak{a}] + 2\mathfrak{a}}{4}, \qquad f \mapsto \frac{[\mathfrak{a}, \mathfrak{h}] + 2\mathfrak{a}}{4}, \qquad h \mapsto \mathfrak{h}.$$

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The elements $\mathfrak{a}, \mathfrak{h}, [\mathfrak{a}, \mathfrak{h}]$ form a basis for $\mathfrak{sl}_2$. We call $\mathfrak{a}, \mathfrak{h}$ the *alternate generators* for $\mathfrak{sl}_2$.

THE $\mathcal{ABH}$-PRESENTATION OF $L(\mathfrak{sl}_2)$

We found that $L(\mathfrak{sl}_2)$ is isomorphic to the Lie algebra over $\mathbb{F}$ that has generators $\mathcal{A}, \mathcal{B}, \mathcal{H}$ and relations

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$\mathcal{A} \mapsto e \otimes 1 + f \otimes 1, \quad \mathcal{B} \mapsto e \otimes t + f \otimes t^{-1}, \quad \mathcal{H} \mapsto h \otimes 1.$

Back to tridiagonal pairs...

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In view of the $\mathcal{ABH}$-presentation of $L(\mathfrak{sl}_2)$ we make the following definition.

Definition

For a tridiagonal pair (A, B) on V that has Krawtchouk type, an element $H \in \mathrm{End}(V)$ is said to be *compatible* with A, B whenever the following relations hold:

$$\begin{aligned} [H,[A,B]] &= 0, \\ [A,[A,H]] &= 4H, \\ [H,[H,A]] &= 4A, \\ [B,[B,H]] &= 4H, \\ [H,[H,B]] &= 4B. \end{aligned}$$

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(So any given compatible element gives a way to extend the tridiagonal pair to a $L(\mathfrak{sl}_2)$-module structure on the underlying vector space. We will state this relationship more precisely shortly.)

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In this talk we will give a description of the set $\mathrm{Com}(A, B)$, but we will focus on a special case in which the results are particularly nice.

SPECIAL CASE: $\rho_i = \binom{d}{i}$

Suppose (A, B) is a tridiagonal pair of Krawtchouk type such that $\rho_i = \binom{d}{i}$ for $0 \le i \le d$. Observe that $\dim(V) = 2^d$ in this case. We can prove the following:

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- The common eigenspaces of these elements all have dimension 1. Let the set X consist of these common eigenspaces, and note that X has cardinality 2^d.

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- There exists a d-cube structure on X with the following property: for all $y \in X$, Ay and By are contained in the sum of those elements of X adjacent to y.
- For all $y \in X$ there exists $H_y \in \mathrm{Com}(A, B)$ such that for $0 \le i \le d$ the sum of the elements in X at distance i from y is an eigenspace for H_y with eigenvalue $d - 2i$. Thus $\mathrm{Com}(A, B)$ equals $\{H_y \mid y \in X\}$.

Finite-dimensional irreducible modules for $L(\mathfrak{sl}_2)$

In order to proceed with our description, we first need to recall the classification of the finite-dimensional irreducible modules for $L(\mathfrak{sl}_2)$. This classification is well-known, and we summarize it now.

Finite-dimensional irreducible modules for $L(\mathfrak{sl}_2)$

In order to proceed with our description, we first need to recall the classification of the finite-dimensional irreducible modules for $L(\mathfrak{sl}_2)$. This classification is well-known, and we summarize it now.

Up to isomorphism, there exists a unique irreducible $L(\mathfrak{sl}_2)$-module of dimension 1. On this module $\mathcal{A}, \mathcal{B}, \mathcal{H}$ each act as the zero map. We call this the *trivial* $L(\mathfrak{sl}_2)$-module.

For nonzero $a \in \mathbb{F}$ we define the Lie algebra homomorphism $EV_a : L(\mathfrak{sl}_2) \to \mathfrak{sl}_2$ by

$$EV_a(u \otimes g(t)) = g(a)u, \quad u \in \mathfrak{sl}_2, \quad g(t) \in \mathbb{F}[t, t^{-1}].$$

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Let V denote an irreducible $\mathfrak{sl}_2$-module with finite dimension at least 2. We pull back by EV_a to get a $L(\mathfrak{sl}_2)$-module structure on V. We call this an *evaluation module* for $L(\mathfrak{sl}_2)$ and denote it by $V(a)$. The $L(\mathfrak{sl}_2)$-module $V(a)$ is irreducible.

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The $L(\mathfrak{sl}_2)$-modules $V(a)$ and $V(b)$ are isomorphic if and only if $a = b$.

Example

Let V be the irreducible $\mathfrak{sl}_2$-module of dimension 2 (the natural module) and consider the evaluation $L(\mathfrak{sl}_2)$-module $V(a)$. The actions of $\mathcal{A}, \mathcal{B}, \mathcal{H}$ on $V(a)$ are given by

$$\mathcal{A} : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{B} : \begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix}, \quad \mathcal{H} : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let U, W denote $L(\mathfrak{sl}_2)$-modules. Then $U \otimes W$ has an $L(\mathfrak{sl}_2)$-module structure given by

$$x.(u \otimes w) = (x.u) \otimes w + u \otimes (x.w), \qquad x \in L(\mathfrak{sl}_2), \quad u \in U, \quad w \in W.$$

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The classification of finite-dimensional irreducible $L(\mathfrak{sl}_2)$-modules is given in the following theorem.

Theorem

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A tensor product of evaluation modules

$$V_1(a_1) \otimes \cdots \otimes V_n(a_n)$$

is irreducible if and only if $a_1, a_2, \ldots, a_N$ are mutually distinct.

Definition

Let V denote a nontrivial finite-dimensional irreducible $L(\mathfrak{sl}_2)$-module. Then V is isomorphic to a tensor product of evaluation modules, say

$$V_1(a_1) \otimes \cdots \otimes V_n(a_n).$$

V is said to be *inverse-free* whenever $a_i \neq a_j^{-1}$ for $1 \leq i, j \leq n$.

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We can now state the relationship between compatible elements and $L(\mathfrak{sl}_2)$-modules more precisely.

Theorem

Let (A, B) be a tridiagonal pair on V of Krawtchouk type, and let $H \in \mathrm{Com}(A, B)$. Then there exists a unique $L(\mathfrak{sl}_2)$-module structure on V such that $\mathcal{A}, \mathcal{B}, \mathcal{H}$ act as A, B, H, respectively. This $L(\mathfrak{sl}_2)$-module is irreducible and inverse-free.

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Theorem

Given an irreducible $L(\mathfrak{sl}_2)$-module structure on V that is inverse-free, $\mathcal{A}, \mathcal{B}$ act on V as a tridiagonal pair of Krawtchouk type and the action of $\mathcal{H}$ is compatible with the actions of $\mathcal{A}, \mathcal{B}$.

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Suppose (A, B) is a tridiagonal pair of Krawtchouk type such that $\rho_i = \binom{d}{i}$ for $0 \leq i \leq d$. Fix a compatible element H. We will now describe all the other compatible elements using H.

By a simple linear algebraic argument, the condition $\rho_i = \binom{d}{i}$ for $0 \leq i \leq d$ just means that each tensor factor in the decomposition referred to above is 2-dimensional.

That is, we can identify V with an irreducible, inverse-free $L(\mathfrak{sl}_2)$-module

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Definition

For $1 \le i \le d$ let $\mathcal{H}_i \in \mathrm{End}(V)$ be

$$I \otimes \cdots \otimes I \otimes \mathcal{H} \otimes I \otimes \cdots \otimes I$$

where $\mathcal{H}$ above is acting on the ith tensor factor.

It is easy to check that the maps $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_d$ are linearly independent and diagonalizable and that they mutually commute.

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Observe that our compatible element $H = \mathcal{H}_1 + \mathcal{H}_2 + \cdots + \mathcal{H}_d$.

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The compatible elements can be described as follows.

Theorem

The set $\mathrm{Com}(A, B)$ consists of the elements

$$\sum_{i=1}^{d} \varepsilon_i \mathcal{H}_i \qquad (\varepsilon_i \in \{\pm 1\},\ 1 \le i \le d).$$

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So far we have the following:

- The set $\mathrm{Com}(A, B)$ has cardinality 2^d.
- The elements of $\mathrm{Com}(A, B)$ mutually commute and are diagonalizable on V.
- The common eigenspaces of these elements all have dimension 1. Let the set X consist of these common eigenspaces, and note that X has cardinality 2^d.

A d-cube structure on X

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- Recall, from an earlier example, that for $0 \leq i \leq d$ the actions of $\mathcal{A}, \mathcal{B}, \mathcal{H}$ on $V_i(a_i)$ are given by

$$\mathcal{A} : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{B} : \begin{pmatrix} 0 & a_i \\ a_i^{-1} & 0 \end{pmatrix}, \quad \mathcal{H} : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

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  with respect to a suitable basis.
- Tensoring such bases together for each $V_i(a_i)$ gives a basis for V that is composed of common eigenvectors for the elements of $\mathrm{Com}(A, B)$.

A d-cube structure on X

- We can therefore identify X with the set of sequences $y = (y_1, y_2, \ldots, y_d)$ such that $y_i \in \{0, 1\}$ for $0 \leq i \leq d$. Here $y_i = 0$ (respectively $y_i = 1$) corresponds to choosing an eigenvector for $\mathcal{H}$ in $V_i(a_i)$ with eigenvalue 1 (respectively eigenvalue -1.)

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- The set X has the following d-cube structure: for $x, y \in X$ we say x is adjacent to y if and only if they differ in exactly one coordinate.

For $y = (y_1, y_2, \ldots, y_d) \in X$ we define $H_y \in \mathrm{End}(V)$ by

$$H_y = \sum_{i=1}^{d} (-1)^{y_i} \, \mathcal{H}_i.$$

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By our description above we have $\mathrm{Com}(A, B) = \{H_y \mid y \in X\}$.

Now we define matrices α, η and β_i for $0 \leq i \leq d$ as follows:

$$\alpha : \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta_i : \begin{pmatrix} 0 & a_i \\ a_i^{-1} & 0 \end{pmatrix}, \quad \eta : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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One can check that the matrices representing A, B and H_y for $y \in X$ are

1. $A = \sum_{i=1}^{d} I^{\otimes(i-1)} \otimes \alpha \otimes I^{\otimes(d-i)}$,
2. $B = \sum_{i=1}^{d} I^{\otimes(i-1)} \otimes \beta_i \otimes I^{\otimes(d-i)}$,
3. $H_y = \sum_{i=1}^{d}(-1)^{y_i} \ I^{\otimes(i-1)} \otimes \eta \otimes I^{\otimes(d-i)}$,

where the tensors above denote the Kronecker product of matrices, and I denotes the two-by-two identity matrix.

By straightforward computations the above equations yield the following.

1. The matrix representing A is the adjacency matrix for the d-cube structure on X.
2. The matrix representing B is a weighted adjacency matrix for the d-cube structure on X.
3. Fix $y \in X$. The matrix representing H_y is the diagonal matrix with (x, x)-entry equal to $d - 2i$ for all $x \in X$ at distance i from y.

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From this it follows that

- for all $y \in X$, Ay and By are contained in the sum of those elements of X adjacent to y, and
- for all $y \in X$ and $0 \leq i \leq d$ the sum of the elements in X at distance i from y is an eigenspace for H_y with eigenvalue $d - 2i$.