1 Problems

1. Use \texttt{ChineseRem} to find (if possible) the smallest solution of the following congruences:

\[
\begin{align*}
    x &\equiv 3 \mod 10, \\
    x &\equiv 8 \mod 15, \\
    x &\equiv 5 \mod 84 \\
    x &\equiv 29 \mod 52, \\
    x &\equiv 19 \mod 72.
\end{align*}
\]

2. Compute the gcd of 42823 and 6409. Find \(x, y \in \mathbb{Z}\) such that

\[
gcd(5033464705, 3138740337) = 5033464705x + 3138740337y.
\]

3. Describe the following sequence: \(a_1 = 3, a_{n+1} = 3a_n \mod 100\).

4. Use \texttt{Product} to compute \(2 \cdot 4 \cdot 6 \cdots 20\).

5. Prove that

\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{9999} - \frac{1}{10000} = \frac{1}{5001} + \frac{1}{5002} + \cdots + \frac{1}{10000}.
\]

6. Find the last two digits of \(3^{400}\).

7. Find the roots of \(x^2 + x + 7 \equiv 0 \mod m\) for \(m \in \{15, 189\}\).

8. Write a function that returns the binary expansion of an integer. Could you do this for other bases?

9. Use \texttt{MinimalPolynomial} to compute the minimal polynomial of \(3 + \sqrt{5}\) over the rational numbers.

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10. Compute the first 100 Fibonacci numbers.

11. For $k \in \mathbb{N}$ let $a_n$ be given by $a_1 = \cdots = a_{k+1} = 1$ and $a_n = a_{n-k} + a_{n-k+1}$ for all $n > k+1$. Write a function depending on $k$ that constructs the sequence $a_n$. For more information see http://oeis.org/A103379.

12 (Somos sequence). Write a function that returns the $n$-th term of $a_n$, where $a_0 = a_1 = a_2 = a_3 = 1$ and

$$a_n = \frac{a_{n-1}a_{n-3} + a_{n-2}^2}{a_{n-4}}$$

for all $n \geq 4$. For more information see http://oeis.org/A006720.

13. Write a function that given a list `lst` of words and a letter $x$, returns a sublist of `lst` where every word starts with $x$.

14. Write a function that returns the number of prime numbers $\leq n$.

15. Use the function `permuted` to write a function that shows all the anagrams of a given word.

16. Given a list of non-negative numbers, write a function that displays the histogram associated with this list. For example, if the argument is the list `[1, 4, 2]`, the function should display

```
X
xxxxx
xx
```

17. Write a function that given a list of words returns the longest one.

18. Write a function that transforms a given number of seconds in days, hours, minutes and seconds.

19. Write a function that returns the average value of a given list of numbers.

20. Write a function that, given a letter, returns `true` if the letter is a vowel and `false` otherwise.

21. Use `CharacteristicPolynomial` to compute the characteristic polynomial of the matrix

$$
\begin{pmatrix}
0 & -1 & 1 \\
1 & 2 & -1 \\
1 & 1 & 0
\end{pmatrix}
$$

Can you compute the minimal polynomial?

22. Use the function `QuotientRemainder` to compute the quotient and the remainder of $f = 2x^4 + 3x^3 + 2x + 4$ and $g = 3x^2 + x + 2$ in the ring $\mathbb{Z}_5[x]$.

23. Compute $3x^{101} - 15x^{16} - 2x^7 - 5x^4 + 3x^3 + 2x^2 + 1 \mod x^3 + 1$. 
24. Prove that $x = 2$ is the only root in $\mathbb{Z}_5$ of $x^{1000} + 4x + 1 \in \mathbb{Z}_5[x]$.

25. Factorize $x^4 - 1$ in $\mathbb{Z}/5[x]$ and in $\mathbb{Z}/7[x]$.

26. Prove that $x^2 - 79x + 1601$ gives a prime number for $x \in \{0, 1, \ldots, 79\}$.

27. Write the first 50 twin primes.

28. FRACTRAN is a programming language invented by J. Conway. A FRACTRAN program is simply an ordered list of positive rationals together with an initial positive integer input $n$. The program is run by updating the integer $n$ as follows:

- For the first rational $f$ in the list for which $nf \in \mathbb{Z}$, replace $n$ by $nf$.
- Repeat this rule until no rational in the list produces an integer when multiplied by $n$, then stop.

Write an implementation of the FRACTRAN language.

Starting with $n = 2$, the program

$\frac{17}{65}, \frac{133}{34}, \frac{17}{19}, \frac{23}{23}, \frac{2233}{31}, \frac{74}{31}, \frac{41}{129}, \frac{13}{41}, \frac{1}{13}$

produces the sequence

$2, 15, 825, 725, 1925, 2275, 425, 390, 330, 290, 770 \ldots$

In 1987, J. Conway proved that this sequence contains the set $\{2^p : p \text{ prime}\}$. See https://oeis.org/A007542 for more information.

29. The first terms of Conway’s “look and say” sequence are the following:

1
11
21
1211
111221
312211

After guessing how each term is computed, write a script to create the first terms of the sequence.

30. Write

$\left(\begin{array}{c} 123456 \\ 253461 \end{array}\right), \left(\begin{array}{c} 123456789 \\ 234517896 \end{array}\right), \left(\begin{array}{c} 12345 \\ 32451 \end{array}\right)$

as a product of disjoint cycles.

31. Write the permutations $(123)(45)(1625)(341)$ and $(12)(245)(12)$ as product of disjoint cycles.
32. Find a permutation $\tau$ such that
1. $\tau(12)(34)\tau^{-1} = (56)(13)$.
2. $\tau(123)(78)\tau^{-1} = (257)(13)$.
3. $\tau(12)(34)(567)\tau^{-1} = (18)(23)(456)$.

33. Compute $\tau \sigma \tau^{-1}$ in the following cases:
1. $\sigma = (123)$ and $\tau = (34)$.
2. $\sigma = (567)$ and $\tau = (12)(34)$.

34. Let $\sigma \in S_9$ be given by $\sigma(i) = 10 - i$ for all $i \in \{1, \ldots, 9\}$. Write $\sigma$ as a product of disjoint cycles.

35. Find (if possible) three permutations $\alpha, \beta, \gamma \in S_5$ such that $\alpha \beta = \beta \alpha$, $\beta \gamma = \gamma \beta$ and $\alpha \gamma \neq \gamma \alpha$.

36. Use the function `PermutationMat` to write the elements of $S_3$ as $3 \times 3$ matrices.

37. For $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 2 & 3 \end{pmatrix}$ compute
$$I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \frac{1}{4!} A^4.$$ 

38. Write the function
$$(n, A) \mapsto I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots + \frac{1}{n!} A^n.$$ 

39. For $n \in \mathbb{N}$ the Hilbert matrix $H_n$ is defined as
$$(H_n)_{ij} = \frac{1}{i+j-1}, \quad i, j \in \{1, \ldots, n\}.$$ 

Write the function $n \mapsto H_n$.

40. Use the function `KroneckerProduct` to compute
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} 5 & 6 & 7 \\ 2 & 1 & 0 \\ 0 & 1 & 9 \end{pmatrix}.$$ 

41. Let $S$ be the vector space (over the rationals) generated by $(0, 1, 0)$ and $(0, 0, 1)$ and $T$ be generated by $(1, 2, 0)$ and $(3, 1, 2)$. Use `VectorSpace` to create these vector spaces and compute $\dim S$, $\dim T$, $\dim (S \cap T)$ and $\dim (S + T)$.

42. Write the coordinates of the vector $(1, 0, 1)$ in the basis given by $(2i, 1, 0)$, $(2, -i, 1)$, $(0, 1+i, 1-i)$.
43. Walsh matrices $H(2^k)$, $k \geq 1$, are defined recursively as follows:

$$H(2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad H(2^k) = H(2) \otimes H(2^{k-1}), \quad k \geq 1,$$

Construct the function $n \mapsto H(2^n)$.

44. Use the functions `Eigenvalues` and `Eigenvalues` to compute the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{pmatrix} \in \mathbb{Q}^{3 \times 3}.$$

The function `Eigenvalues` returns generators of the eigenspaces, where $v \neq 0$ is an eigenvector of $A$ with eigenvalue $\lambda$ if and only if $vA = \lambda v$.

45. Use the function `NullspaceMat` to compute the nullspace of the matrix $A$ from problem 44.

The nullspace of $A$ is defined as the set of vectors $v$ such that $vA = 0$.

46. Compute the order of the group generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Can you recognize this group?

47. Construct the Heisenberg group $H(\mathbb{Z}/3)$.

48. Construct the Klein group as a group of permutations.

49. Prove that all subgroups of $C_4 \times Q_8$ are normal.

50. Let $G$ be the set of matrices of the form

$$\begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix}, \quad c, d \in \mathbb{F}_4, \quad c \neq 0.$$

Prove that $G$ is a group and compute its order.

51. Use the function `IsomorphicSubgroups` to prove that $A_6$ does not contain a subgroup isomorphic to $S_5$ and that $A_7$ contains a subgroup isomorphic to $S_5$.

52. Prove that $A_6$ does not contain subgroups of prime index.

53. Prove that $SL_2(3)$ has a unique normal subgroup of order eight.

54. Find a subgroup of $SL_2(5)$ isomorphic to $SL_2(3)$. 
55. Use the functions SylowSubgroup and ConjugacyClassSubgroups to construct all Sylow subgroups of $A_4$ and $S_4$.

56. Prove that $S_5$ has a 2-Sylow subgroup isomorphic to the dihedral group of eight elements.

57. Can you recognize the structure of 2-Sylow subgroups of $S_6$?

58. Use the function Normalizer to compute the number of conjugates of 2-Sylow subgroups of $A_5$.

59. Find all Sylow subgroups of $C_{27}$, $SL_2(5)$, $S_7$, $S_3 \times A_4$ and $S_3 \times C_{20}$.

60. Compute the conjugacy classes of subgroups of $S_3 \times S_3$ and find three $p$-Sylow subgroups, say $A, B, C$, such that $A \cap B = 1$ and $A \cap C \neq 1$.

61. Prove that the group
$$\{ \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} : b, d \in \mathbb{F}_{19}, d \neq 0 \}$$
is not simple.

62. Let $G$ be the group generated by the permutations $(12)(611)(812)(913), (5139)(61011)(7812)$ and $(2,4,3)(5,8,9)(61013)(71112)$. How many elements of $G$ are commutators?

63. Prove that $A_4 \times C_7$ does not contain subgroups of index two.

64. Prove that $A_5$ does not contain subgroups of order 8, 15, 20, 24, 30, 40.

65. It is known that an abelian subgroup of $S_n$ has order $\leq 3^{\lfloor n/3 \rfloor}$. How good is this bound? For $n \in \{5, 6, 7, 8\}$ find (if possible) an abelian subgroup of $S_n$ of order $3^{\lfloor n/3 \rfloor}$.

66. Prove that $SL_2(5)$ does not contain subgroups isomorphic to $A_5$.

67. Prove that for each $d$ that divides 24 there exists a subgroup of $S_4$ of order $d$.

68. Prove that $SL_2(3)$ contains a unique element of order two. Prove that $SL_2(3)$ does not have subgroups of order 12.

69. Prove that the derived subgroup of $SL_2(3)$ is isomorphic to $Q_8$.

70. Can you recognize the group $SL_2(3)/Z(SL_2(3))$?

71. Are the groups $S_5$ and $SL_2(5)$ isomorphic?

72. Let $\langle r^4 = s^2 = 1, srs = r^{-1} \rangle$ be the dihedral group of eight elements. Find all subgroups containing $\langle 1, r^2 \rangle$.

73. Find all the group homomorphisms $S_3 \rightarrow SL_2(3)$. 
74. Are there any surjective homomorphism $D_{16} \to D_8$? What about $D_{16} \to C_2$?

75. Prove that $\text{Aut}(A_4) \cong S_4$.

76. Prove that $\text{Aut}(D_8) \cong D_8$ and that $\text{Aut}(D_{16}) \not\cong D_{16}$.

77. Compute the order of the group $\text{Aut}(C_{11} \times C_2 \times C_3)$.

78. Prove that $D_{12} \cong S_3 \times C_2$.

79. Prove that every group of order $< 60$ is solvable.

80. Let $G$ be a group of order twelve such that $G \not\cong A_4$. Prove that $G$ contains an element of order six.

81. Prove that a group of order 455 is cyclic.

82. Let $G$ be a simple group of order 168. Compute the number of elements of order seven of $G$.

83. Prove that there are no simple groups of order 2540 and 9075.

84. Find a group $G$ of order $3^6$ such that $\{[x,y] : x,y \in G\} \neq [G,G]$.

85. Find a group $G$ of order $2^7$ such that $\{[x,y] : x,y \in G\} \neq [G,G]$.

86. Prove that a group of order 15, 35 or 77 is cyclic.

87. Prove that a simple group of order 60 is isomorphic to $A_5$.

88. Prove that the only non-abelian simple group of order $< 100$ is $A_5$.

89. Is the following true? For any finite group $G$ the set $\{x^2 : x \in G\}$ is a subgroup of $G$.

90. Prove the following theorem of Guralnick [1]. There exists a group $G$ of order $n \leq 200$ such that $[G,G] \neq \{[x,y] : x,y \in G\}$ if and only if $n \in \{96,128,144,162,168,192\}$.

91. Prove the following extension of Guralnick’s theorem (Problem 90). There exists a group $G$ of order $n < 1024$ such that $[G,G] \neq \{[x,y] : x,y \in G\}$ if and only if $n$ is one of the following numbers: 96, 128, 144, 162, 168, 192, 216, 240, 256, 270, 288, 312, 320, 324, 336, 360, 378, 384, 400, 432, 448, 450, 456, 480, 486, 504, 512, 528, 540, 560, 576, 594, 600, 624, 640, 648, 672, 702, 704, 720, 729, 744, 750, 756, 768, 784, 792, 800, 810, 816, 832, 840, 864, 880, 882, 888, 896, 900, 912, 918, 936, 960, 972, 1000, 1008.

92. Compute the list of normal subgroups of $GL_2(3)$.

93. Compute the list of minimal subgroups of $A_4$. 
94. Compute the socle and the list of minimal normal subgroups of $A_4$.

95. Compute the Fitting and the Frattini subgroup of $\text{SL}_2(3)$.

96. Compute the list of all maximal normal subgroups of $\text{SL}_2(3)$.

97. Prove that $\text{PSL}_2(7)$ has a maximal subgroup of order 16.

98. Let $G$ be a finite group and $H$ be a subgroup. The Chermak–Delgado measure of $H$ is the number $m_G(H) = |H| |C_G(H)|$. Write a function to compute the Chermak–Delgado.

99. Compute $m_G(H)$ for $G \in \{S_3, D_8\}$ and $H$ a subgroup of $G$.

100. Compute order the holomorph of $A_4$. Find a permutation representation of small degree and find some minimal normal subgroup of order four. This is an exercise of [2].

101. Prove that the group $\langle (123\cdots 7), (26)(34) \rangle$ is simple, has order 168 and acts transitively on $\{1,\ldots, 7\}$.

102. Compute the order of the group $\langle a, b : a^2 = b^2 = (bab^{-1})^3 = 1 \rangle$.

103. Prove that $\langle a, b : a^2 = aba^{-1}b = 1 \rangle$ is an infinite group.

104. Compute the order of the group $\langle a, b : a^8 = b^2a^4 = ab^{-1}ab = 1 \rangle$.

105. Can you recognize the group $\langle a, b : a^5 = 1, b^2 = (ab)^3, (a^3ba^4b)^2 = 1 \rangle$?

106. Prove that the group $\langle a, b : a^2 = b^3 = a^{-1}b^{-1}ab = 1 \rangle$ is finite and cyclic.

107. Prove that the group $\langle a, b : a^2 = b^3 = 1 \rangle$ is non-abelian.

108. Compute the order of the group $\langle a, b, c : a^3 = b^3 = c^3 = 1, aba = bab, cbc = bcb, ac = ca \rangle$.

109. Prove that the group $\langle a, b, c : bab^{-1} = a^2, cbc^{-1} = b, aca^{-1} = c^2 \rangle$ is trivial. This is an exercise of Serre’s book [3, §1]:

110. Prove that $B(2, 3)$ is isomorphic to the Heisenberg group $H(\mathbb{Z}/3)$.

111. Find a permutation representation of the group $B(2, 3)$.

112. Prove that $B(3, 3)$ is a finite group of order $\leq 2187$.

113. Let $G$ be a finite group with $k$ conjugacy classes. It is known that the probability that two elements of $G$ commute is equal to $\text{prob}(G) = k/|G|$. Compute this probability for $\text{SL}_2(3), \mathbb{A}_4, \mathbb{S}_4$ and $Q_8$. 
2 Some solutions

1 For the first system one obtains that $x = 173$ is the smallest solution such that $x \in \{0, 1, \ldots, 420\}$:

\[
gap> \text{Lcm}(10, 15, 84);
420
\]
\[
gap> \text{ChineseRem}([10, 15, 84],[3, 8, 5]);
173
\]

Since the other system does not have solutions, the function \texttt{ChineseRem} returns an error message:

\[
gap> \text{ChineseRem}([52, 72],[29, 19]);
\text{Error, the residues must be equal modulo 4 called from}
\text{<function "ChineseRem">( <arguments> )}
called from read-eval loop at line 149 of *stdin*
you can 'quit;' to quit to outer loop, or
you can 'return;' to continue
brk>
\]

2

\[
gap> \text{Gcd}(42823, 6409);
17
\]
\[
gap> \text{GcdRepresentation}(5033464705, 3137640337);
[ 107535067, -172509882 ]
\]

3 The sequence is 3, 27, 87, 87, 87, .... Here is the code:

\[
gap> n := 10;;
gap> s := [3];;
gap> for k in [1..n] do
> Add(s, 3^s[k] \mod 100);
> od;
gap> s;
[ 3, 27, 87, 87, 87, 87, 87, 87, 87, 87, 87 ]
\]

4

\[
gap> \text{Product([2,4..20])};
3715891200
\]

5 It is worth noting that the exercise is a particular case of a general formula. However, here is the code to solve this particular exercise:

\[
gap> n := 5000;;
gap> \text{Sum(List([1..2*n], j->(-1)^(j+1)*1/j))} =
> \text{Sum(List([n+1..2*n], j->1/j))};
true
\]

Yet another way to do this, probably less elegant, is
Leandro Vendramin

6

gap> 3^400;
70550791086553325712464271575934796216507949612787315\ 76287122320292620855515829341565792985294713415815495\ 23348253559118669297930718245666941450844545352570279\ 60285323760313192443283334088001

7 There are no solutions of $x^2 + x + 7 \equiv 0 \mod 15$:

gap> List(Filtered(Integers mod 15, x->IsZero(x^2+x+7)), Int);
[ ]
gap> List(Filtered(Integers mod 189, x->IsZero(x^2+x+7)), Int);
[ 13, 49, 76, 112, 139, 175 ]

12

a:=function(n);
> if n in [0,1,2,3] then
> return 1;
> else
> return (a(n-1)*a(n-3)+a(n-2)^2)/a(n-4);
> fi;
> end;
function( n ) ... end

14

gap> f:=function(n);
> s:=0;
> for x in [1..n] do
> if IsPrime(x)=true then
> s:=s+1;
> fi;
> od;
> return s;
> end;
function( n ) ... end
16

gap> f:=function(x)
> local n,k;
> for n in [1..Number(x)] do
> for k in [1..x[n]] do
> Print("X");
> od;
> Print("\n");
> od;
> end;
function( x ) ... end

21 To compute the minimal polynomial one uses \texttt{MinimalPolynomial}:

gap> m := [[0,-1,1],[1,2,-1],[1,1,0]];;
gap> CharacteristicPolynomial(m); x^3-2*x^2+x
gap> MinimalPolynomial(m); x^2-x

22 The quotient is $-x^2 + 3x + 3$ and the remainder is $3x + 3$:

gap> x := Indeterminate(GF(5));;
gap> f := 2*x^4+3*x^3+2*x+4;;
gap> g := 3*x^2+x+2;;
gap> QuotientRemainder(f,g);
[ -x_1^2+Z(5)^3*x_1+Z(5)^3, Z(5)^3*x_1+Z(5)^3 ]

23 The answer is $-x^2 + 18x - 2$:

gap> x := Indeterminate(Rationals);;
gap> (3*x^101-15*x^16-2*x^7-5*x^4+3*x^3+2*x^2+1) mod (x^3+1);
-x^2+18*x-2

24

gap> x := Indeterminate(GF(5));;
gap> RootsOfPolynomial(x^1000+4*x+1);
[ Z(5) ]
gap> 2*Z(5)^0 = Z(5);
true

25

gap> x := Indeterminate(Integers mod 5);;
gap> Factors(x^4-1);
[ x_1+Z(5)^0, x_1+Z(5), x_1-Z(5)^0, x_1+Z(5)^3 ]
gap> x := Indeterminate(Integers mod 7);;
gap> Factors(x^4-1);
[ x_1+Z(7)^0, x_1-Z(7)^0, x_1^2+Z(7)^0 ]

26 Let us check that all these eighty numbers are primes.
gap> Filtered(List([0..79], x->x^2-79*x+1601), IsPrime);
[ 1601, 1523, 1447, 1373, 1301, 1231, 1163, 1097, 1033, 971,
  911, 853, 797, 743, 691, 641, 593, 547, 503, 461, 421, 383,
  347, 313, 281, 251, 223, 197, 173, 151, 131, 113, 97, 83,
  71, 61, 53, 47, 43, 41, 43, 47, 53, 61, 71, 83, 97,
  113, 127, 151, 173, 197, 223, 251, 281, 313, 347, 383, 421,
  461, 503, 547, 593, 641, 691, 743, 797, 853, 911, 971,
  1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601 ]
gap> Length(last);
80

27 Let us compute the first 50 twin primes:

gap> l := [];;
gap> n := 2;;
gap> repeat
  > if IsPrime(n) and IsPrime(n+2) then
    > Add(l, [n,n+2]);
  > fi;
  > n := n+1;
until Size(l)=50;
gap> l;
[ [ 3, 5 ], [ 5, 7 ], [ 11, 13 ], [ 17, 19 ], [ 29, 31 ],
  [ 41, 43 ], [ 59, 61 ], [ 71, 73 ], [ 101, 103 ],
  [ 107, 109 ], [ 137, 139 ], [ 149, 151 ], [ 179, 181 ],
  [ 191, 193 ], [ 197, 199 ], [ 227, 229 ], [ 239, 241 ],
  [ 269, 271 ], [ 281, 283 ], [ 311, 313 ], [ 347, 349 ],
  [ 419, 421 ], [ 431, 433 ], [ 461, 463 ], [ 521, 523 ],
  [ 569, 571 ], [ 599, 601 ], [ 617, 619 ], [ 641, 643 ],
  [ 659, 661 ], [ 809, 811 ], [ 821, 823 ], [ 827, 829 ],
  [ 857, 859 ], [ 881, 883 ], [ 1019, 1021 ], [ 1031, 1033 ],
  [ 1049, 1051 ], [ 1061, 1063 ], [ 1091, 1093 ],
  [ 1151, 1153 ], [ 1229, 1231 ], [ 1277, 1279 ],
  [ 1289, 1291 ], [ 1301, 1303 ], [ 1319, 1321 ],
  [ 1427, 1429 ], [ 1451, 1453 ], [ 1481, 1483 ],
  [ 1487, 1489 ] ]

28

gap> fractran := function(n, lst, bound)
  > local i, j, seq;
  > seq := [n];
  > for i in [1..bound] do
    > for j in [1..Size(lst)] do
      > if IsInt(lst[j]*n) then
        > n := lst[j]*n;
        > Add(seq, n);
      > break;
    > fi;
  > od;
  > od;
  > return seq;
  > end;

27 Let us compute the first 50 twin primes:

gap> l := [];;
gap> n := 2;;
gap> repeat
  > if IsPrime(n) and IsPrime(n+2) then
    > Add(l, [n,n+2]);
  > fi;
  > n := n+1;
until Size(l)=50;
gap> l;
[ [ 3, 5 ], [ 5, 7 ], [ 11, 13 ], [ 17, 19 ], [ 29, 31 ],
  [ 41, 43 ], [ 59, 61 ], [ 71, 73 ], [ 101, 103 ],
  [ 107, 109 ], [ 137, 139 ], [ 149, 151 ], [ 179, 181 ],
  [ 191, 193 ], [ 197, 199 ], [ 227, 229 ], [ 239, 241 ],
  [ 269, 271 ], [ 281, 283 ], [ 311, 313 ], [ 347, 349 ],
  [ 419, 421 ], [ 431, 433 ], [ 461, 463 ], [ 521, 523 ],
  [ 569, 571 ], [ 599, 601 ], [ 617, 619 ], [ 641, 643 ],
  [ 659, 661 ], [ 809, 811 ], [ 821, 823 ], [ 827, 829 ],
  [ 857, 859 ], [ 881, 883 ], [ 1019, 1021 ], [ 1031, 1033 ],
  [ 1049, 1051 ], [ 1061, 1063 ], [ 1091, 1093 ],
  [ 1151, 1153 ], [ 1229, 1231 ], [ 1277, 1279 ],
  [ 1289, 1291 ], [ 1301, 1303 ], [ 1319, 1321 ],
  [ 1427, 1429 ], [ 1451, 1453 ], [ 1481, 1483 ],
  [ 1487, 1489 ] ]

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gap> fractran := function(n, lst, bound)
  > local i, j, seq;
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  > for i in [1..bound] do
    > for j in [1..Size(lst)] do
      > if IsInt(lst[j]*n) then
        > n := lst[j]*n;
        > Add(seq, n);
      > break;
    > fi;
  > od;
  > od;
  > return seq;
  > end;
function( n, lst, bound ) ... end

gap> code := [17/91, 78/85, 19/51, 23/38, 29/33, 77/29, 95/23, 
  77/19, 1/17, 11/13, 13/11, 15/2, 1/7, 55/1];;
gap> fractran(2, code, 10);
[ 2, 15, 825, 725, 1925, 2275, 425, 390, 330, 290, 770 ]

41 We know that \( \dim(S + T) = \dim S + \dim T - \dim(S \cap T) \). To compute \( S \cap T \) we use Intersection. Dimensions of vector spaces are computed with Dimension:

\[
\text{gap> S := VectorSpace(Rationals, } \begin{bmatrix}0,1,0\end{bmatrix}, \begin{bmatrix}0,0,1\end{bmatrix});\;
\text{gap> T := VectorSpace(Rationals, } \begin{bmatrix}1,2,0\end{bmatrix}, \begin{bmatrix}3,1,2\end{bmatrix});\;
\text{gap> Dimension(S); 2}
gap> \text{Dimension(T); 2}
gap> \text{Dimension(Intersection(S,T)); 1}
\]

Now \( \dim(S + T) = \dim S + \dim T - \dim(S \cap T) = 2 + 2 - 1 = 3 \).

42 Recall that \( i \in E(4) \). First we need to create the vector space \( V \) over the smallest field containing \( E(4) \); this is done with Field(E(4)). With Basis we create the linear basis of \( V \) and compute the coordinates of \( [1,0,1] \) with the function Coefficients:

\[
\text{gap> lst := } \begin{bmatrix}2*E(4),1,0\end{bmatrix}, \begin{bmatrix}2,-E(4),1\end{bmatrix}, \begin{bmatrix}0,1+E(4),1-E(4)\end{bmatrix};\;
\text{gap> V := VectorSpace(Field(E(4)), lst);};
\text{gap> Coefficients(Basis(V, lst), [1,0,1]);}
[ 0, 1/2, 1/4+1/4*E(4) ]
\]

51

\[
\text{gap> IsomorphicSubgroups(AlternatingGroup(6),SymmetricGroup(5)); [ ]}
\text{gap> IsomorphicSubgroups(AlternatingGroup(7),SymmetricGroup(5)); [ [ (1,2,3,4,5), (1,2) ] -> [ (3,4,5,6,7), (1,2)(3,4) ] ]}
\]

53

\[
\text{gap> Number(NormalSubgroups(SL(2,3)), x->Order(x)=8); 1}
\]

54 One can easily verify that

\[
\left\langle \begin{pmatrix}0 & 2 \\ 2 & 4\end{pmatrix}, \begin{pmatrix}1 & 3 \\ 0 & 2\end{pmatrix} \right\rangle \cong \text{SL}_2(3).
\]

Let us see how is that we found these matrices. One possible approach requires the use of the function IsomorphicSubgroups:

\[
\text{gap> l := IsomorphicSubgroups(SL(2,5), SL(2,3));}
\text{gap> gr := Image(l[1]);}
\]
Alternatively, we could try to find all subgroups that are isomorphic to $\text{SL}_2(3)$. To save some time, we compute conjugacy classes of subgroups instead of all possible subgroups. At the end, we see that there is only one conjugacy class of subgroups isomorphic to $\text{SL}_2(3)$ and this conjugacy class has 5 elements:

```
gap> f := Filtered(ConjugacyClassesSubgroups(SL(2,5)), \x->IdGroup(Representative(x))=IdGroup(SL(2,3)))
group([ [ [ Z(5)^2, 0*Z(5) ], [ 0*Z(5), Z(5)^2 ] ], [ 0*Z(5), Z(5) ], [ Z(5), 0*Z(5) ] ], [ [ 0*Z(5), Z(5)^2 ], [ Z(5)^0, 0*Z(5) ] ], [ [ Z(5)^0, Z(5) ], [ Z(5)^0, Z(5)^3 ] ] ])^G
```

```
gap> Size(f[1])
5
```

64 It is enough to study the orders of the representatives of conjugacy classes of subgroups.

```
gap> A5 := AlternatingGroup(5);;
gap> c := ConjugacyClassesSubgroups(A5);;
gap> Intersection([8,15,20,24,30,40], \List(c, x->Size(Representative(x))))

[ ]
```

75

```
gap> A4 := AlternatingGroup(4);;
gap> StructureDescription(AutomorphismGroup(A4))
"S4"
```

97 The group $\text{PSL}_2(7)$ has order 2448, so the following approach will work:

```
gap> 16 in List(MaximalSubgroups(PSL(2,17)), Order);
true
```

Alternatively, one can compute conjugacy classes of maximal subgroups. This approach will be better for groups of bigger order.

```
gap> List(ConjugacyClassesMaximalSubgroups(PSL(2,17)), \x->Order(Representative(x)))
[ 136, 24, 24, 16, 16 ]
gap> 16 in last;
true
```
gap> ChermakDelgado := function(group, subgroup)
>      return Size(subgroup) * Size(Centralizer(group, subgroup));
>  end;

Recall that the holomorph of $A_4$ is the group $\text{Aut}(A_4) \rtimes A_4$.

gap> A4 := AlternatingGroup(4);
gap> hol := SemidirectProduct(AutomorphismGroup(A4), A4);
gap> f := IsomorphismPermGroup(Image(f));
gap> G := Image(f);
gap> Order(G);
288
gap> MovedPoints(G);
[ 1, 2, 3, 4, 5, 6, 7, 8 ]
gap> MinimalNormalSubgroups(G);
[ Group([ (1,2)(3,4), (1,4)(2,3) ]), Group([ (5,7)
(6,8), (5,8)(6,7) ] ) ]
gap> List(last, Order);
[ 4, 4 ]
gap> GeneratorsOfGroup(G);
[ (3,4)(7,8), (2,4,3)(6,8,7), (1,2)(3,4)(5,6)(7,8),
(1,3)(2,4)(5,7)(6,8), (6,8,7), (5,7)(6,8),
(5,6)(7,8) ]

gap> G := f/[a^8, b^2*a^4, a*b^-1*a*b];
gap> Order(G);
16

gap> f := FreeGroup(2);
gap> a := f.1;;
gap> b := f.2;;
gap> G := f/[a^5, b^2*Inverse((a*b)^3), (a^3*b*a^4*b)^2];
gap> Order(G);
60
gap> StructureDescription(G);
"A5"
108

```gap
> f := FreeGroup(3);;
> a := f.1;;
> b := f.2;;
> c := f.3;;
> G := f/[a*b*a*Inverse(b*a*b), a*c*Inverse(c*a), b*c*b*Inverse(c*b*c), a^3, b^3, c^3];;
> Order(G);
648
> StructureDescription(G);
"((C3 x C3) : C3) : Q8) : C3"
```

109

```gap
> f := FreeGroup(3);;/D
> a := f.1;;
> b := f.2;;
> c := f.3;;
> gr := f/[b*a*b^(-1)*a^(-2), c*b*c^(-1)*b^(-2), a*c*a^(-1)*c^(-2)];;
> IsTrivial(gr);
true
```

112

```gap
> f := FreeGroup(3);;
> rels := Set(List([1..10000], x->Random(f)ˆ3));;
> B33 := f/rels;;
> Order(B33);
2187
```

113

```gap
> probability := function(group)
> > return Size(ConjugacyClasses(group))/Order(group);
> end;
> function( group ) ... end
> gap> probability(SL(2,3));
7/24
> gap> probability(AlternatingGroup(4));
1/3
> gap> probability(AlternatingGroup(5));
1/12
> gap> probability(SymmetricGroup(4));
5/24
> gap> probability(QuaternionGroup(8));
5/8
```
References