Making Measurements in Physics

1. Introduction

No measurements are exact. Since in the physics laboratory you will be performing measurements you need to know how to assign experimental uncertainties to them. Moreover, you will need to know how to write your final results correctly, how to analyze your data or interpret your graphs as well as compare your results with accepted values or to each other. Finally, you are required to evaluate your work and come up with possible shortcomings of the experimental procedure and/or equipment i.e., to find possible sources of uncertainty. The following pages contain the information you will need to do all that.

2. Types of uncertainties

**Random uncertainties** – express random variations between individual measurements of the same quantity caused by factors difficult or impossible to control and the **precision** of the measuring instrument. Those uncertainties can never be eliminated i.e. are never equal to zero however the smaller their numerical value is the more precise the measurement is. It is your responsibility to identify and track random uncertainties during the lab.

Sources: Most often the measuring instrument, which has been manufactured to perform measurements with certain precision, is the primary source of random uncertainty. Its value is often indicated in the user manual. For simple instruments like rulers, triple beam balances or any device with a scale, it depends on the smallest division on that scale, as well as your ability to read the appropriate measurement. For example, a meter stick is usually accurate to 1 mm, however, not all objects can be measured accurately to that precision. For digital instruments, you can take the $\pm 1$ on the last decimal place shown on the display as your uncertainty.

- **Instrument precision** (meter stick, triple beam balance, stopwatch)

**Systematic uncertainties** – express the deviation of a measurement from an accepted (or true) value and represent the **accuracy** of your measurements. The bigger they are the less accurate your measurement is. It is not always possible to obtain a numerical value for this type of uncertainty, however it is often possible to identify causes for which your measured value differs from the accepted one.

Sources: Any process which affects your measurements in a systematic way i.e. makes all of your measurements performed with a specific device larger or smaller. Examples include:

- **Air resistance or friction** (consistently slowing things down)
- **Temperature** (affects the speed of sound or length of objects)
- **Calibration**
- **Parallax** – could be present while using an instrument with a scale however should be avoided in the laboratory.
Below you can see a visual representation of the concepts of accuracy and precision and how they are related to experimental uncertainties. The bull’s eye represents the true or accepted value of a quantity.

<table>
<thead>
<tr>
<th>Precise not accurate</th>
<th>Accurate not precise</th>
<th>precise and accurate</th>
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</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Image 1" /></td>
<td><img src="image2.png" alt="Image 2" /></td>
<td><img src="image3.png" alt="Image 3" /></td>
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<tr>
<td>systematic – high random - low</td>
<td>systematic – low random – high</td>
<td>systematic – low random - low</td>
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### 3. Three ways to Represent Random Uncertainties

- **Absolute uncertainty**, $\delta x$, is a number representing precision of your measurement. If $x$ represents measured quantity then $\delta x$ (delta $x$) represents its absolute uncertainty. It always has the same units as $x$.

- **Relative uncertainty**, $\frac{\delta x}{x}$, is a number which allows one to judge the relative size of the absolute uncertainty $\delta x$ and the measured quantity $x$. It’s obtained by dividing absolute uncertainty by measured value.

- **Percent uncertainty**, $\frac{\delta x}{x} \times 100\%$ is a relative uncertainty expressed as a percentage.

Example: If you measured a mass of an object to be 3.4 kg with a precision of 0.3 kg then:

$$x = 3.4 \text{ kg}; \quad \delta x = 0.3 \text{ kg}; \quad \frac{\delta x}{x} = 0.09; \quad \text{or 9\%.}$$

### 4. Writing down your measurements

There is a specific way to write down your final result together with the associated absolute uncertainty which we will use in the lab. Suppose we have

$$x = (3.4127 \pm 0.0003) \text{ kg}$$

or if you use scientific notation

$$x = (3412.7 \pm 0.3) \times 10^{-3} \text{ kg}$$
Note the use of brackets in the above example, i.e. the fact that the units and the common multiplier \( \times 10^{-3} \) are outside of it since they apply to both numbers. To correctly write your result this way you must use the following rule:

*Write down your absolute uncertainty with only one significant figure (that’s the 3 in the example above), and then write down your measured (or calculated) value with the same number of decimal places as you ended up with in the uncertainty.*

Below are some rules which will allow you to identify the number of significant figures:

- all non-zero digits are always significant
- all zeros between non-zero digits are always significant
- Leading zeros are not significant (0.002 has only 1 sig. fig.)
- Trailing zeros in a number with a decimal point are significant (0.0230 has 3 sig. figures “2”, “3” and “0”)
- When a number ends with zeros which are to the left of the decimal eg. 19,000 we cannot tell if those zeros are significant or not. The number has to be rewritten using scientific notation i.e. if you know that the first zero is significant then you would write 1.90 \( \times \) \( 10^4 \). Now we have the case of the trailing zero (see previous rule) to the right of the decimal therefore it is significant.

Finally you need to know how many significant digits to keep when you perform mathematical operations on numbers with different number of sig. figs.

**Addition/Subtraction**

The number of decimal places after addition or subtraction is equal to the smallest number of decimal places in any of the individual terms.

**Multiplication/Division**

The number of significant figures after multiplication or division is equal to the smallest number of significant figures in any of the individual terms.

*Note:* The only difference in the wording of these two rules is that in the addition/subtraction rule uses *decimal places* whereas the multiplication/division rule uses *significant figures*.

**5. Multiple measurements**

Sometimes you will report an average value as your experimental result. In this section we explain how to calculate an absolute uncertainty of that average. It is not estimated as in the case of a single measurement.
It is important to know that when you are collecting data for averaging you do not have to estimate the absolute uncertainty for each individual measurement. You use statistics to calculate that quantity. In particular it is the \textit{standard deviation} which is an estimate of the absolute uncertainty for each measurement in case of multiple measurements. It gives an indication about how dispersed or spread out the measurements are.

The formula is:

\[
\sigma_x = \sqrt{\frac{\sum(x_i - \bar{x})^2}{N - 1}}
\]

where \(\sigma_x\) (sigma \(x\)) is a symbol for \textit{standard deviation}, \(x_i\) is each individual measurement, \(\bar{x}\) is the average and \(N\) is the number of measurements. To calculate the \textit{uncertainty of the average} (\textit{standard error}) you simply have to divide the standard deviation by the square root of the number of measurements i.e.

\[
\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}
\]

\textbf{Let's take a look at an example.}

Below is a list of steps one needs to go through to find experimental uncertainty of an average. There is also an example in table form. You should follow along the table in the order of the numbered steps.

1. Take multiple measurements.
2. Calculate the average value of the measurements.
3. Find the deviation of each measurement from the mean.
4. Square each deviation, add the squares, and divide by this \(N - 1\).
5. Take the square root of this answer. You now have the standard deviation.
6. Divide by \(\sqrt{N}\), where \(N\) is the number of measurements.
7. Example: here is an example of 4 measurements that were taken for the length of an object.
| $L_i$ [cm] | $|L_i - \bar{L}|$ [cm] | $(|L_i - \bar{L}|)^2$ [cm$^2$] |
|-----------|-----------------|-------------------------------|
| 27.7      | 0.2             | 0.04                          |
| 27.1      | 0.4             | 0.16                          |
| 27.8      | 0.3             | 0.09                          |
| 27.5      | 0.0             | 0.00                          |
| $\bar{L} = 27.5$ | $\sum (|L_i - \bar{L}|)^2 = 0.29$ | $\sigma_L = \sqrt{\frac{\sum (|L_i - \bar{L}|)^2}{N-1}} = 0.3 \text{ cm}$ |
|           |                 | $\sigma_L = \frac{\sigma_L}{\sqrt{N}} = 0.2 \text{ cm}$ |

Therefore the final results is: $\bar{L} = (27.5 \pm 0.2) \text{ cm}$

6. Experimental Uncertainty in a Calculated Result (Rules)

When you do a calculation using quantities that have experimental uncertainties, the resulting quantity must also have an experimental uncertainty. To find it there are rules for different operations that must be used.

**Addition/Subtraction Rule**

When finding a result by adding or subtracting uncertain quantities, the absolute experimental uncertainty in the result is the sum of the absolute experimental uncertainties of the uncertain quantities.

$$IF \ z = x + y \ OR \ z = x - y \ THEN$$

$$\delta z = \delta x + \delta y$$

Example:

$$A = (122 \pm 5) \text{ m}, \quad B = (207 \pm 2) \text{ m}$$

$$C = A + B = 122 \text{ m} + 207 \text{ m} = 329 \text{ m}$$

$$\delta C = \delta A + \delta B = 5 \text{ m} + 2 \text{ m} = 7 \text{ m}$$

$$C = (329 \pm 7) \text{ m}$$
**Multiplication/Division Rule**

When finding a result by multiplying or dividing uncertain quantities, the relative experimental uncertainty in the result is the sum of the relative experimental uncertainties of the uncertain quantities.

\[
IF \ z = x \times y \ OR \ z = \frac{x}{y} \ THEN
\]

\[
\frac{\delta z}{z} = \frac{\delta x}{x} + \frac{\delta y}{y}
\]

**Example:**

\[
A = (120 \pm 5) m, \ B = (200 \pm 10) m
\]
\[
C = A \times B = 120 \times 200 m = 24000 m^2
\]
\[
\frac{\delta C}{C} = \frac{\delta A}{A} + \frac{\delta B}{B} = \frac{5 m}{120 m} + \frac{10 m}{200 m} = 0.0917
\]
\[
\delta C = \frac{\delta C}{C} \times C = 0.0917 \times 24000 m^2 = 2200 m^2
\]
\[
C = (24000 \pm 2000) m^2 \ or \ (2.4 \pm 0.2) \times 10^4 m^2
\]

**Note:** The uncertainty \(\delta C = 2200 m^2\) has been rounded to one significant figure, so it becomes \(2000 m^2\).

**Important:** the only difference in the two rules is one uses absolute experimental uncertainty and one uses relative experimental uncertainty. If you use the multiplication rule, you can always get the absolute experimental uncertainty from the relative experimental uncertainty.

**Exponent Rule (derived from Multiplication/Division)**

\[
IF \ z = x^n = x \times x \times ... \times x \ THEN
\]

\[
\frac{\delta z}{z} = n \frac{\delta x}{x}
\]
Multiplication by a Constant (Shortcut Rule derived from Multiplication/Division)

\[ \text{IF } z = ax \text{ THEN } \delta z = a \delta x \]

Combination of Rules
These first three rules can be used in combination.

Example: find the density of a ball with radius \( r = 0.1246 \pm 0.0002 \) m and a mass of \( m = 2.35 \pm 0.04 \) kg and its uncertainty.

1. First calculate \( \rho \):
   \[ \rho = \frac{m}{\frac{4}{3} \pi r^3} = \frac{\frac{2.35}{3} \pi}{(0.1246 m)^3} = 290.018 \text{ kg/m}^3 \]

2. Uncertainty formula comes from applying multiplication/division and exponent rules
   \[ \frac{\delta \rho}{\rho} = \frac{\delta m}{m} + 3 \frac{\delta r}{r} = \frac{0.04 \text{ kg}}{2.35 \text{ kg}} + 3 \frac{(0.0002 \text{ m})}{(0.1246 \text{ m})} = 0.0218 \]
   \[ \delta \rho = \frac{\delta \rho}{\rho} \times \rho = \pm 6 \text{ kg/m}^3 \]

3. Write quantity with experimental uncertainty:
   \[ \rho = (2.90 \pm 0.06) \times 10^2 \text{ kg/m}^3 \]

7. Comparing Two Results
If you have a value \( x \) and an uncertainty \( \delta x \) then the results agree with some expected value \( x_{\exp} \)
only if they agree within experimental uncertainty. This means that the value \( x_{\exp} \) is between \( x - \delta x \)
and \( x + \delta x \).
Example: say you have determined that a speed of sound in air is $v_1 = (338 \pm 9) \text{m/s}$. The accepted value however is $v = 343 \text{m/s}$. According to your measurement you can tell that the speed of sound is somewhere in the range $v_1 - \delta v_1$ and $v_1 + \delta v_1$ i.e. $329 \text{m/s} \leq v_1 \leq 347 \text{m/s}$. Since 343 m/s falls inside that range you can tell that your measurement agrees with the accepted value within experimental uncertainty.

If another person obtained $v_2 = 333 \pm 7 \text{m/s}$ as his result then, in order to compare the two results, i.e. $v_1$ and $v_2$ you have to calculate both ranges. Therefore, $329 \text{m/s} \leq v_1 \leq 347 \text{m/s}$ and $326 \text{m/s} \leq v_2 \leq 340 \text{m/s}$ and since the two ranges overlap we can say that the two results agree with each other within an experimental uncertainty. However, you can see that the second result does not agree with the accepted value since 343 m/s does not fall within its range.

Remember that you will not lose marks if your results do not agree within the experimental uncertainty between each other or with the accepted value as long as it is clear from your data that you performed the measurements carefully and correctly.