Physics 3820. Mathematical Physics II
Final Examination

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Duration of the examination: 120 minutes. Attempt one question of Part A, one question of Part B and one question of Part C and the question in part D.

Part A (30%)

Problem (1) Evaluate the integral by using the residue theorem:
\[
\int_{0}^{2\pi} \frac{\cos \theta}{5 + 4 \cos \theta} \, d\theta
\]

Problem (2) Evaluate the integral by using the residue theorem:
\[
\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 25)(x^2 + 16)} \, dx
\]

Part B (30%)

Problem (1) Find a solution of the equation:
\[
(1 - x^2) \frac{d^2 y}{dx^2} + y = 0
\]
as a series in powers of \( x \). Find the recurrence relation and write the first five terms of the series.

Problem (2)
Find the two solutions of the Bessel equation:
\[
6x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x - 1)y = 0
\]
as a Frobenius series in power of \( x \). Find the recurrence relations for the two independent solutions and write the first three terms of the series for each independent solution.

Part C (30%)

Problem (1) Find the Fourier transform of the function
\[
f(t) = \frac{t}{t^2 + a^2}
\]

Problem (2) An electron in an atom may be modeled classically as a damped harmonic oscillator:
\[
\frac{d^2 x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega_0^2 x = f(t)
\]
The electron is driven by an incoming EM wave with electric field

\[ E(t) = \begin{cases} \quad E_0 e^{-\alpha t} \sin(\Omega t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \]

(a) What is the appropriate \( f(t) \) for this problem

(b) Solve for the transform \( x(\omega) \) of the electron’s position.

**Hint:** Use the relation:

\[ \sin \beta = \frac{e^{i\beta} - e^{-i\beta}}{2i} \]

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**Part D (10%)**

**Problem (1)**

(a) Find the Fourier transform \( X(\omega) \) of the solution of the equation:

\[ \frac{d^2x}{dt^2} + \omega_0^2 x = f(t) \]

(b) Write the inverse Fourier transform of \( X(\omega) \) and therefore write an integral expression about the solution \( x(t) \).

(c) Write down the form of the Green’s function \( G(t - t') \) solution:

\[ x(t) = \int_{-\infty}^{\infty} f(t')G(t - t')dt' \]

Find the expression for the Green’s function in integral form, but without evaluating the integral of the complex integrand.
FORMULAE

(1) The Taylor series. Suppose \( f(z) \) is analytic in a region \( R : |z - a| \leq \rho \). Then the series:

\[
f(z) = f(a) + (z - a)f'(a) + \frac{(z - a)^2}{2} f''(a) + \ldots + \frac{(z - a)^n}{n!} \frac{d^n f}{dz^n} \bigg|_{z=a} + \ldots
\]

is uniformly convergent within the circle \( |z - a| \leq \rho \), where \( \rho \) is the radius of convergence.

(2) The Laurent series. Suppose \( f(z) \) is analytic in an angular region \( R : \rho_1 < |z - a| < \rho_2 \). Then

\[
f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n, \quad \text{where} \quad c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}} d\xi \quad -\infty < n < \infty
\]

(3) The order of a pole \( z = a \) is the lowest integer \( p \) for which the limit \( \lim_{z \to a} (z - a)^p f(z) \) exists.

(4) The Residue Theorem: If a function \( f \) is analytic in a simply connected domain \( D \) except for finite number of isolated singularities and if curve \( C \) is within \( D \), then:

\[
\oint_C f dz = 2\pi i \sum_{n=1}^{N} \text{Res} f(z_n)
\]

where \( z_n \) are singularities of \( f \) within \( C \).

(5) Finding Residues:
(i) \( \text{Res}(f(a)) \) is equal to the coefficient \( c_{-1} \) of the Laurent series at \( z = a \).
(ii) For a simple pole:

\[
\text{Res} f(a) = \lim_{z \to a} (z - a)f(z)
\]

(iii) For a pole of order \( m \):

\[
\text{Res} f(a) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - a)^m f(z)]
\]

(iv) For a function of the form \( f(z) = g(z)/h(z) \), where \( h(z) \) has a simple zero at \( z = a \) and \( g(z) \) is analytic at \( a \):

\[
\text{Res} f(a) = \lim_{z \to a} \frac{g(z)}{h'(z)}
\]
(6) **Jordan's Lemma:** If $f(z)$ converges uniformly to zero wherever $z \to \infty$, then
\[
\lim_{R \to \infty} \int_{C_R} f(z) e^{ikz} dz = 0
\]
where
(i) $C_R$ is the upper half of the circle $|z| = R$ when $k$ is positive and
(ii) $C_R$ is the lower half of the circle $|z| = R$ when $k$ is negative.

(7) **Fourier Series:** Periodical function $f(x)$ with a period $L$, may be expressed as
(a) Real Fourier series:
\[
f(x) = \sum_{n=0}^{n=\infty} a_n \sin \left( \frac{2n\pi x}{L} \right) + b_n \cos \left( \frac{2n\pi x}{L} \right)
\]
where
\[
a_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{2n\pi x}{L} \right) dx
\]
\[
b_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{2n\pi x}{L} \right) dx
\]
\[
b_0 = \frac{1}{L} \int_0^L f(x) dx
\]
(b)
\[
f(z) = \sum_{n=0}^{n=\infty} c_n e^{i\left( \frac{2n\pi x}{L} \right)}
\]
where
\[
c_n = \frac{2}{L} \int_0^L f(x) e^{-i\left( \frac{2n\pi x}{L} \right)} dx \quad \text{and} \quad c_0 = \frac{1}{L} \int_0^L f(x) dx
\]

(8) **Fourier transform.**
We defined in class two forms of Fourier transform, which are equivalent. The both forms may be used in solving physical problems:
(a) The Fourier transform of the function $f(t)$ is
\[
F(\omega) = \mathcal{F}(f(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt
\]
The inverse Fourier transform of $F(\omega)$ is:
\[
f(t) = \mathcal{F}^{-1}(F(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega
\]
(b) When solving physical problems related to waves propagation, often the Fourier transform is defined as:
\[
F(\omega) = \mathcal{F}(f(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt
\]
The inverse Fourier transform of $F(\omega)$ in this case is:

$$f(t) = \mathcal{F}^{-1} \{ F(\omega) \} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega$$

(8) Properties of Fourier transform:

(i) Linearity:

$$\mathcal{F} \{ f(t) + g(t) \} = \mathcal{F} \{ f(t) \} + \mathcal{F} \{ g(t) \}$$

$$\mathcal{F} \{ af(t) \} = a \mathcal{F} \{ f(t) \}$$

(ii) Complex Conjugate

$$F^*(\omega) = \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \right]^* = F(-\omega)$$

(iii) Differentiation

(a) If the Fourier transform is defined as in (7a) then:

$$\mathcal{F} \left( \frac{df(t)}{dt} \right) = i\omega \mathcal{F} \{ f(t) \} = i\omega F(\omega)$$

and

$$\mathcal{F} \left( \frac{d^n f(t)}{dt^n} \right) = (i\omega)^n \mathcal{F} \{ f(t) \} = (i\omega)^n F(\omega)$$

(b) If the Fourier transform is defined as in (7b) then:

$$\mathcal{F} \left( \frac{df(t)}{dt} \right) = -i\omega \mathcal{F} \{ f(t) \} = -i\omega F(\omega)$$

and

$$\mathcal{F} \left( \frac{d^n f(t)}{dt^n} \right) = (-i\omega)^n \mathcal{F} \{ f(t) \} = (-i\omega)^n F(\omega)$$

(iv) Attenuation and shifting

$$\mathcal{F} \{ e^{at} f(t) \} = F(\omega + ia)$$

$$\mathcal{F} \{ f(t - a) \} = e^{-ia\omega} F(\omega)$$

(v) Parseval’s Theorem: If $F(\omega) = \mathcal{F} \{ f(t) \}$ and $G(\omega) = \mathcal{F} \{ g(t) \}$ then

$$\int_{-\infty}^{\infty} f(x)g(x)dx = \int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega$$

and therefore:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$
(vi) Convolution: If $F(\omega) = \mathcal{F}(f(t))$ and $G(\omega) = \mathcal{F}(g(t))$ then the inverse Fourier transform:

$$\mathcal{F}^{-1}(F(\omega)G(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)g(x-u)du$$