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Making Measurements in Physics

1. Introduction

No measurements are exact. Since in the physics laboratory you will be performing measurements, you need to know how to assign experimental uncertainties to them. Moreover, you will need to know how to write your final results correctly, how to analyze your data or interpret your graphs as well as compare your results with accepted values or to each other. Finally, you are required to evaluate your work and come up with possible shortcomings of the experimental procedure and/or equipment i.e., to find possible sources of uncertainty. The following pages contain the information you will need to do all that.

2. Types of uncertainties

Random uncertainties – express random variations between individual measurements of the same quantity caused by factors difficult or impossible to control and the _precision_ of the measuring instrument. Those uncertainties can never be eliminated, i.e., are never equal to zero. However, the _smaller_ the numerical value is, the _more precise_ the measurement is. It is your responsibility to identify and track random uncertainties during the lab.

Sources: Most often the measuring instrument, which has been manufactured to perform measurements with certain precision, is the primary source of random uncertainty. Its value is often indicated in the user manual. For simple instruments like rulers, triple beam balances or any device with a scale, it depends on the smallest division on that scale. For example, a meter stick is accurate up to 1 mm or 0.1 cm (0.001 m). For digital instruments, you can take the ± 1 on the last decimal place shown on the display as your uncertainty.

Systematic uncertainties – express the deviation of a measurement from an accepted (or true) value and represent the _accuracy_ of your measurements. The _bigger_ they are, the _less accurate_ your measurement is. It is not always possible to obtain a numerical value for this type of uncertainty, however it is often possible to identify causes for which your measured value differs from the accepted one.

Sources: Any process which affects your measurements in a systematic way, i.e. makes all of your measurements performed with a specific device larger or smaller. Examples include:

- _Air resistance or friction_ (consistently slowing things down)
- _Temperature_ (affects the speed of sound or length of objects)
- _Calibration_

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- *Parallax* – could be present while using an instrument with a scale however should be avoided in the laboratory.

Below you can see a visual representation of the concepts of *accuracy* and *precision* and how they are related to *experimental uncertainties*. The bull's eye represents the true or accepted value of a quantity.

Precise not accurate	Accurate not precise	precise and accurate
systematic – high random - low	systematic – low random – high	systematic – low random - low
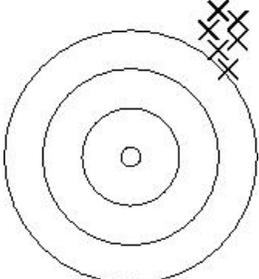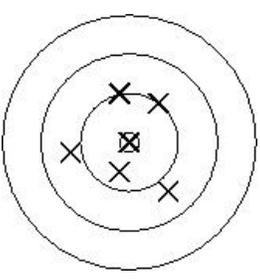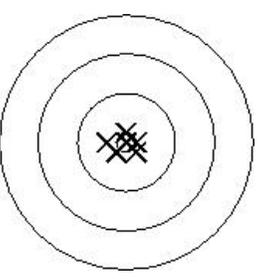
3. The ways to Represent Random Uncertainties

Absolute uncertainty, δx, is a number representing precision of your measurement. If x represents measured quantity, then δx (delta x) represents its absolute uncertainty. δx always has the same units as x.

Relative uncertainty, $\frac{\delta x}{x}$, is a number which allows one to judge the relative size of the absolute uncertainty δx and the measured quantity x. It's obtained by dividing absolute uncertainty δx by measured value x. The relative uncertainty is unitless.

The uncertainty (absolute or relative) in this lab, is always rounded to 1 significant figure.

Example: If you measured a mass of an object to be 3.4127 kg with a precision of 0.0003 kg, then:

$$x = 3.4127 \; kg; \;\; \delta x = 0.0003 \; kg; \;\; \frac{\delta x}{x} = 0.00009.$$

4. Writing down your measurements

There is a specific way to write down your final result together with the associated absolute uncertainty which we will use in the lab.

$$x \pm \delta x$$

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Suppose we have

$$x = (3.4127 \pm 0.0003) \; kg$$

or if you use scientific notation

$$x = (3412.7 \pm 0.3) \times 10^{-3} \; kg$$

Note: the use of brackets in the above example, i.e. the fact that the units and the common multiplier 10^{-3} are outside of it since they apply to both numbers.

To correctly write your result this way you must use the following rule:

Write down your absolute uncertainty with only one significant figure (that's the 3 in the example above), and then write down your measured (or calculated) value with the same number of decimal places as you ended up with in the uncertainty.

Below are some rules which will allow you to identify the number of significant figures:

- all non-zero digits are always significant.
- all zeros between non-zero digits are always significant.
- Leading zeros are not significant ($0\,002$ has only 1 significant figure).
- Trailing zeros in a number with a decimal point are significant ($0\,0230$ has 3 sig. figures "2", "3" and "0").
- When a number ends with zeros which are to the left of the decimal, eg. $19\,000$ we cannot tell if those zeros are significant or not. The number has to be rewritten using scientific notation, i.e. if you know that the first zero is significant then you would write 1.9×10^4. Now we have the case of the trailing zero (see previous rule) to the right of the decimal therefore it is significant.

Finally you need to know how many significant digits to keep when you perform mathematical operations on numbers with different number of significant figures.

Addition/Subtraction

The number of decimal places after addition or subtraction is equal to the smallest number of decimal places in any of the individual terms.

Multiplication/Division

The number of significant figures after multiplication or division is equal to the smallest number of significant figures in any of the individual terms.

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Note: The only difference in the wording of these two rules is that in the addition/subtraction rule uses *decimal places* whereas the multiplication/division rule uses *significant figures*

5. Multiple measurements

Sometimes you will report an average value as your experimental result. In this section, we explain how to calculate an absolute uncertainty of that average. It is not estimated as in the case of a single measurement.

It is important to know that when you are collecting data for averaging, you do not have to estimate the absolute uncertainty for each individual measurement. You use statistics to calculate that quantity. In particular, it is the ***standard deviation*** which is an estimate of the absolute uncertainty for each measurement in case of multiple measurements. It gives an indication about how dispersed or spread out the measurements are.

The formula is:

$$\sigma_x = \sqrt{\frac{\sum(x_i - \bar{x})^2}{N - 1}}$$

where σ_x (sigma x) is a symbol for *standard deviation*, x_i is each individual measurement, $\bar{x}$ is the average and N is the number of measurements. To calculate the *uncertainty of the average (standard error),* you simply have to divide the standard deviation by the square root of the number of measurements, i.e.

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$

Let's take a look at an example.

Below is a list of steps one needs to go through to find experimental uncertainty of an average. There is also an example in table form. You should follow along the table in the order of the numbered steps.

1. Take multiple measurements x_i.
2. Calculate the average value $\bar{x}$ of the measurements.
3. Find the deviation of each measurement from the mean $|x_i - \bar{x}|$.
4. Square each deviation, add the squares, and divide by this $N - 1$.
5. Take the square root of this answer. You now have the standard deviation σ_x.
6. Divide σ_x by $\sqrt{N}$ to calculate the standard error $\sigma_{\bar{x}}$, where N is the number

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of measurements.
7. Write the measurement value in the form of $\bar{x} \pm \sigma_{\bar{x}}$.

Example: Here is an example of 4 measurements that were taken for the length of an object.

L_i [cm]	$|L_i - \bar{L}|$ [cm]	$(|L_i - \bar{L}|)^2$ [cm²]
27.7	0.175	0.030625
27.1	0.425	0.180625
27.8	0.275	0.075625
27.5	0.025	0.000625
$\bar{L} = 27.525$		$\sum(|L_i - \bar{L}|)^2 = 0.2875$

$$\sigma_L = \sqrt{\frac{\sum(L_i - \bar{L})^2}{N - 1}} = 0.31\ cm$$

$$\sigma_{\bar{L}} = \frac{\sigma_L}{\sqrt{N}} = 0.16\ cm$$

Therefore the final results is: $\bar{L} = (27.5 \pm 0.2)\ cm$

6. Experimental Uncertainty in a Calculated Result (Rules)

When you do a calculation using quantities that have experimental uncertainties, the resulting quantity must also have an experimental uncertainty. To find it there are rules for different operations that must be used.

Addition/Subtraction Rule

When finding a result by adding or subtracting uncertain quantities, the absolute experimental uncertainty in the result is the sum of the absolute experimental uncertainties of the uncertain quantities.

$$IF\ \ z = x + y\ \ OR\ \ z = x - y\ \ THEN$$

$$\delta z = \delta x + \delta y$$

Example: Given $A = (122 \pm 5)\ m$, $B = (207 \pm 2)\ m$, find $C = A + B$

$$C = A + B = 122\ m + 207\ m = 329\ m$$
$$\delta C = \delta A + \delta B = 5\ m + 2\ m = 7\ m$$
$$C = (329 \pm 7)\ m$$

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Multiplication/Division Rule

When finding a result by multiplying or dividing uncertain quantities, the relative experimental uncertainty in the result is the sum of the relative experimental uncertainties of the uncertain quantities.

$$IF \quad \pmb{z = x \times y} \quad OR \quad \pmb{z = \frac{x}{y}} \quad THEN$$

$$\frac{\pmb{\delta z}}{\pmb{z}} = \frac{\pmb{\delta x}}{\pmb{x}} + \frac{\pmb{\delta y}}{\pmb{y}}$$

Example: Given $A = (120 \pm 5)m, \quad B = (200 \pm 10)m$, find $C = A + B$

$$C = A \times B = 120 \; m \times 200 \; m = 24000 \; m^2$$
$$\frac{\delta C}{C} = \frac{\delta A}{A} + \frac{\delta B}{B} = \frac{5 \; m}{120 \; m} + \frac{10 \; m}{200 \; m} = 0.0917$$
$$\delta C = \frac{\delta C}{C} \times C = 0.0917 \times 24000 \; m^2 = 2200 \; m^2$$
$$C = (24000 \pm 2000)m^2 \quad or \quad (2.4 \pm 0.2) \times 10^4 m^2$$

Note: The uncertainty $\delta C = 2200 \; m^2$ has been rounded to one significant figure, so it becomes $2000 \; m^2$.

Important: the only difference in the two rules is one uses absolute experimental uncertainty and one uses relative experimental uncertainty. If you use the multiplication rule, you can always get the absolute experimental uncertainty from the relative experimental uncertainty.

Exponent Rule (derived from Multiplication/Division)

$$IF \quad \pmb{z = x^n} = x \times x \times ... \times x \quad THEN$$

$$\frac{\pmb{\delta z}}{\pmb{z}} = \pmb{n} \frac{\pmb{\delta x}}{\pmb{x}}$$

Note: This works for fractional powers as well.

Example: Given $A = (120 \pm 5) \; m, \quad B = (200 \pm 10) \; m$, find $C = A^2 \times \sqrt[3]{B}$

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$$C = A^2 \times \sqrt[3]{B} = 120^2 \ m^2 \times \sqrt[3]{200} \ \sqrt[3]{m} = 84211 \ m^{\frac{7}{3}}$$

$$\frac{\delta C}{C} = 2\frac{\delta A}{A} + \frac{1}{3}\frac{\delta B}{B} = 2 \times \frac{5 \ m}{120 \ m} + \frac{1}{3} \times \frac{10 \ m}{200 \ m} = 0.1$$

$$\delta C = \frac{\delta C}{C} \times C = 0.1 \times 84211 \ m^{\frac{7}{3}} = 8421.1 \ m^{\frac{7}{3}}$$

$$C = (84000 \pm 8000) \ m^{\frac{7}{3}} \ or \ (8.4 \pm 0.8) \times 10^4 \ m^{\frac{7}{3}}$$

Combination of Rules

These first three rules can be used in combination.

Example: Find the density and its uncertainty of a ball with radius $r = 0.1246 \pm 0.0002 \ m$ and mass $m = 2.35 \pm 0.04 \ kg$.

1. First calculate density ρ:

$$\rho = \frac{m}{V} = \frac{m}{\frac{4}{3}\pi r^3} = \frac{2.35 \ kg}{\frac{4}{3}\pi (0.1246 \ m)^3} = 290.018 \ \frac{kg}{m^3}$$

2. Uncertainty formula comes from applying multiplication/division and exponent rules

$$\frac{\delta \rho}{\rho} = \frac{\delta m}{m} + 3\frac{\delta r}{r} = \frac{0.04 \ kg}{2.35 \ kg} + 3\left(\frac{0.0002 \ m}{0.1246 \ m}\right) = 0.0218$$

$$\delta \rho = \frac{\delta \rho}{\rho} \times \rho = 6.32 \ \frac{kg}{m^3}$$

3. Write quantity with experimental uncertainty:

$$\rho = (290 \pm 6) \ \frac{kg}{m^3} \ or \ \rho = (2.90 \pm 0.06) \times 10^2 \ \frac{kg}{m^3}$$

7. Comparing Two Results

If you have a value x and an uncertainty δx, then the results agree with some expected value x_{exp} only if they agree within experimental uncertainty. This means that the value x_{exp} is between $x - \delta x$ to $x + \delta x$.

Example: Given two values of speed of sound in the air: $v_1 = 338 \pm 9 \ m/s$ and $v_2 = 333 \pm 7 \ m/s$. The expected value of speed of sound is $343 \ m/s$. Do the two values agree with each other within experimental uncertainty? Do the two values agree with the accepted value?

1. Write the range of v_1: $329 \ m/s \leq v_1 \leq 347 \ m/s$.
2. Write the range of v_2: $326 \ m/s \leq v_2 \leq 340 \ m/s$.

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3. Check the overlap. If the ranges overlap, the values agree. If the ranges do not overlap, the values do not agree.

v_1 and v_2 overlap between the range $329\ m/s$ to $340\ m/s$, so v_1 and v_2 agree with each other within the experimental uncertainty. The accepted value $343\ m/s$ falls in the range of v_1, but not v_2, so v_1 agrees with the accepted value while v_2 doesn't agree with the accepted value.

Remember that you will not lose marks if your results do not agree within the experimental uncertainty between each other or with the accepted value, as long as it is clear from your data that you performed the measurements carefully and correctly.

8. Combining answers with uncertainties:

To combine answers with uncertainties, you must follow the rules:

1. List your measured or calculated value and its uncertainty.
2. Start with the uncertainty first.
3. Round the uncertainty to **1 significant figure**.
4. Examine the number. Keep the same number of places after the decimal in the number as you have in the uncertainty.

Example: Automatic fit data gives a slope of $2.7061\ m/s$ with an uncertainty of $0.005611\ m/s$.

1. The slope is $2.7061\ m/s$ and its uncertainty is $0.005611\ m/s$.
2. Start with the uncertainty $0.005611\ m/s$.
3. Rounded the uncertainty to $0.006\ m/s$ (1 significant figure).
4. This uncertainty value has three places after the decimal, so the slope value of 2.7061 is rounded to $2.706\ m/s$.
5. The slope will be $(2.706 \pm 0.006)\ ^m/_s$.