Physics 3820. Final exam

December 14, 2004

E. Demirov

Problem (1) Use subscript notation to verify the following identities:

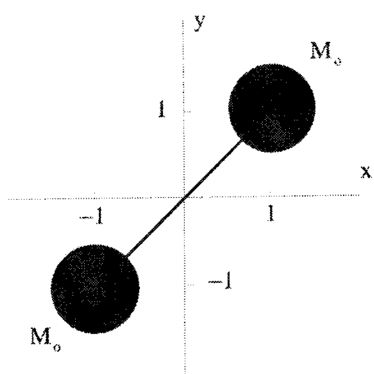
$$(a) \quad \mathbf{A} \times (\nabla \times \mathbf{A}) = \frac{1}{2}\nabla \mathbf{A}^2 - (\mathbf{A} \cdot \nabla)\,\mathbf{A}$$

$$(b) \quad \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

Hint: In problem (1a) you may use the following equality, which we have written in class:

$$\varepsilon_{jki}\,\varepsilon_{jlm} \;=\; \delta_{kl}\,\delta_{im} \;-\; \delta_{km}\,\delta_{il}$$

Problem (2) Consider a dumbbell positioned in xy-plane of a Cartesian system, as shown in the figure below.



The moment of inertia tensor for this object expressed in this Cartesian system is:

$$\mathbf{I} = M_0 \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

(a) Find the basis vectors of a coordinate system in which the inertia tensor is diagonalized;

(b) Find the tensor matrix in the new coordinate system.

Problem (3) Small amplitude waves in a plasma are described by the relations

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n_0 v) = 0$$

$$\varepsilon_0 \frac{\partial E}{\partial x} = -en$$

$$m\frac{\partial v}{\partial t} = -eE - m\nu v$$

where n_0, e, m, ν, and ε_0 are constants. The constant ν is the collision frequency. Assume that n, E, and v are all proportional to $exp(ikx - i\omega t)$.

(a) Solve the equation for nonzero n, E, and v to show that ω satisfies the equation:

$$\omega^2 + i\nu\omega = \frac{n_0 e^2}{m\varepsilon_0} = \omega_p$$

where ω_p is the plasma frequency.

(b) Solve this equation to find the frequency ω and show that the collisions damp the waves.

Problem (4)

(a) Find the Fourier series on the range $0 \le x \le 2\pi$ for the function $f(x) = \sin\ \alpha x$, where α is *not* an integer.

(b) Check your result by evaluating the limit of the Fourier series when $\alpha \to n$.

Hint:

$$\sin(\beta)\sin(\gamma) = \frac{1}{2}\left[\cos(\beta-\gamma) - \cos(\beta+\gamma)\right]$$

$$\sin(\beta)\cos(\gamma) = \frac{1}{2}\left[\sin(\beta+\gamma) + \sin(\beta-\gamma)\right]$$

Problem (5) Find the Fourier transform of the function:

$$f_1(t) = \begin{cases} \sin(\Omega t)\cdot e^{-at} & if \quad t \ge 0 \\ 0 & if \quad t < 0 \end{cases}$$

where $a > 0$.

Hint: Use the relation:

$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$$