THE THIRTY-NINTH W.J. BLUNDON MATHEMATICS CONTEST*

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

February 22, 2023

1. If $3x - y = 10$, what is the value of $\frac{8^x}{2^y}$?
   Solution:
   Since $8 = 2^3$, we have $8^x = 2^{3x} = 2^{y+10} = 2^y\, 2^{10}$. Here we used the relation $3x = y + 10$.
   Thus, $\frac{8^x}{2^y} = 2^{10} = 1024$

   Answer: 1024.

2. Show that $10^{25} - 7$ is divisible by 3.

   Solution:

   $10^{25} - 7 = (10^{25} - 1) - 6 = 9999...9 - 6$ and both summands are divisible by 3.

3. Find $x^2 + y^2$ if x and y are positive integers such that

$$xy + x + y = 71 \quad \text{and} \quad x^2y + xy^2 = 880.$$

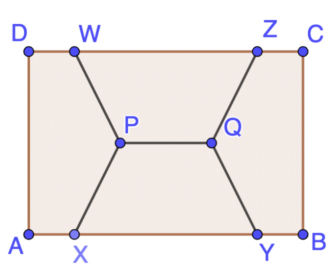
   Solution: Observe that $xy + x + y + 1 = 72$, and so $(x + 1)(y + 1) = 72$. The pairs of integers greater than 1, that multiply to give 72 are: (2,36), (3,24), (4,18), (6,12), (8,9).

   Consequently, the possibilities for pairs (x, y) are (1,35), (2,23), (3,17), (5,11), (7,8).

   For each pair we calculate $x^2y + xy^2 = xy(x + y)$ and find that only the pair $(5, 11)$ gives 880. Finally we calculate $5^2 + 11^2 = 25 + 121 = 146$.

   Answer: $= 146$

4. ABCD is a rectangle divided into four parts of equal area by 5 segments as shown in the figure below.



* A grant in support of this activity was received from the Canadian Mathematical Society.
La Société mathématique du Canada a donné un appui financier à cette activité.

Let PQ and AB be parallel and

$$XY = YB + BC + CZ = ZW = WD + DA + AX$$

If BC=19 cm and PQ=97 cm, find the length of AB in cm.

Solution: WLOG, assume that the picture is symmetric, that is, $YB = ZC = DW = AX = x$ and the segment PQ lies on the horizontal axis of symmetry.

Let E and F be the foots of the perpendicular on AB from P and Q respectively, and $FY = XE = y$.

From the equity of areas of the rectangles $ZCBY$ and $PQFE$, we have $19x = \frac{19}{2} \cdot 97$, so $x = 97/2$.

Then $XY = 97 + 2y = 19 + 2x = 19 + 97$. Then $y = 19/2$.

Then $AB = 97 + 2x + 2y = 97 + 97 + 19 = 213$.

Answer: 213

5. Given that $\cos(2x) = \frac{5}{13}$ find all possible exact values of $\sin x + \cos x$. Express your answer in the form $a\sqrt{b}/c$, where a, b, c are integers.

   Solution: Note that if $\cos(2x) = \frac{5}{13}$ then $\sin(2x) = \pm\sqrt{1 - \frac{25}{169}} = \pm\frac{12}{13}$.

   Let $y = \sin x + \cos x$. Then $y^2 = 1 + \sin(2x) = 1 \pm \frac{12}{13}$.

   If $y^2 = \frac{25}{13}$ then $y = \pm 5\sqrt{13}/13$.

   If $y^2 = \frac{1}{13}$ then $y = \pm\sqrt{13}/13$.

   Answer: $\dfrac{\pm 5\sqrt{13}}{13}$ or $\dfrac{\pm\sqrt{13}}{13}$.

6. A turtle and a boy share the same birthday. For six consecutive birthdays the turtle's age is an integral multiple of the boy's age. How old can the turtle be at the sixth of these birthdays, if it can potentially live up to 200 years? List all the possibilities.

   Solution: One can find by inspection that one of the answers is 66 because

   $$1|61, \quad 2|62, \quad 3|63, \quad 4|64, \quad 5|65, \quad 6|66.$$

   More systematically, let n be the age of the boy on the first of these birthdays and d be the difference in ages. Then we need

   $$(n+k)|(n+k+d), \quad \text{for} \quad k = 0, 1, 2, 3, 4, 5.$$

   Equivalently,

   $$(n+k)|d, \quad \text{for} \quad k = 0, 1, 2, 3, 4, 5.$$

   Let $L = \text{LCM}[n, (n+1), (n+2), (n+3), (n+4), (n+5)]$.

* *A grant in support of this activity was received from the Canadian Mathematical Society.*
La Société mathématique du Canada a donné un appui financier à cette activité.

Then any multiple of L gives a solution.

Take $n = 1$. Then $L = \text{LCM}[1, 2, 3, 4, 5, 6] = 60$. Taking $d = L = 60$, for $k = 5$ we have the age of the turtle to be $1 + 60 + 5 = 66$.

Taking $d = 2L = 120$ we get $1 + 120 + 5 = 126$. We verify that

$$1|121, \quad 2|122, \quad 3|123, \quad 4|124, \quad 5|125, \quad 6|126.$$

Taking $d = 3L = 180$ we get $1 + 180 + 5 = 186$. We verify that

$$1|181, \quad 2|182, \quad 3|183, \quad 4|184, \quad 5|185, \quad 6|186.$$

Taking $d = 4L = 240$ we get $1 + 240 + 5 = 246 > 200$.

If we take $n = 2$, then $L = \text{LCM}[2, 3, 4, 5, 6, 7] = 420 > 200$.

So, there are no more possibilities.

Answer: 66, 126, 186.

7. a) Find a triple of integers (a, b, c) such that $a+b+c = 0$ and $abc = 2$. Calculate $a^3+b^3+c^3$.

   Solution: For example, $a = 2, b = c = -1$; $a^3 + b^3 + c^3 = 8 - 1 - 1 = 6$.

   b) Find a triple of integers (a, b, c) such that $a+b+c = 0$ and $abc = 6$. Calculate $a^3+b^3+c^3$.

   Solution: For example, $a = 3, b = -2, c = -1$; $a^3 + b^3 + c^3 = 27 - 8 - 1 = 18$.

   c) Make a conjecture about the way to find $a^3 + b^3 + c^3$ provided the sum $a + b + c = 0$ and the product $abc = K$, where the value of K is given.

   Solution: $a^3 + b^3 + c^3 = 3K$.

   d) Prove your conjecture.

   Solution: Use $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ so $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$;

   Then $a^3 + b^3 + c^3 = (a + b)^3 + c^3 - 3ab(a + b)$;

   Use $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ with $x = a + b$ and $y = c$ to get

   $(a + b)^3 + c^3 = (a + b + c)((a + b)^2 - (a + b)c + c^2) = 0$ because $a + b + c = 0$.

   As well, $a + b = -c$. Thus, $a^3 + b^3 + c^3 = -3ab(a + b) = 3abc = 3K$.

8. I wrote down three positive numbers. Divided by their range, the product of their mode with the difference of their mean and their median was 6:

$$\frac{(Mean - Median) \times Mode}{Range} = 6.$$

   What was the minimum of the three numbers that I wrote?

   Solution:
   Let x, y, z be the set, ordered from small to large. Thus, y is the median and $z - x$ is the range. Since the product exists and is non-zero, there is a mode (at least two of x, y, z are

* *A grant in support of this activity was received from the Canadian Mathematical Society.*
La Société mathématique du Canada a donné un appui financier à cette activité.

identical) and the median is not equal the mean (not all three are equal). Therefore, our set has the form x, x, z or x, z, z and its mean is

$$\bar{X} = \begin{cases} \frac{2x+z}{3} & \text{(if median is x)} \\ & \text{or} \\ \frac{x+2z}{3} & \text{(if median is z).} \end{cases}$$

The difference, mean − median, is therefore

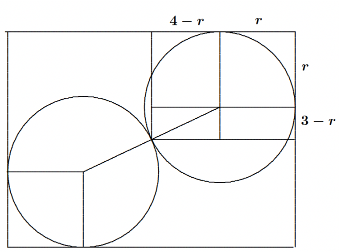
$$\bar{X} - \tilde{X} = \begin{cases} \dfrac{2x+z}{3} - \dfrac{3x}{3} = \dfrac{z-x}{3} & \text{for } x, x, z \\ \dfrac{x+2z}{3} - \dfrac{3z}{3} = \dfrac{x-z}{3} & \text{for } x, z, z \end{cases}$$

For x, x, z, we have $6 = \dfrac{(Mean - Median) \times Mode}{Range} = \dfrac{\frac{z-x}{3} \times x}{z-x} = \dfrac{x}{3}$.
For x, z, z, we have $6 = \dfrac{(Mean - Median) \times Mode}{Range} = \dfrac{\frac{x-z}{3} \times z}{z-x} = \dfrac{-z}{3}$.
Since $z > 0$ and $6 > 0$, the latter case reaches a contradiction: we solve $6 = \dfrac{x}{3}$ and conclude that $x = 18$ is the minimum.

Answer: 18.

9. A rectangle ABCD has sides AB=CD=6 cm and AD=BC=8 cm. Two identical circles are drawn inside of the rectangle such that each circle is tangent to two adjacent sides of the rectangle (AB and BC) and (CD and DA) respectively. In addition, the circles are tangent to each other. Find the exact value (in cm) of the radius of these circles.

Solution:



We identify the right triangle with sides $4 - r, 3 - r$ and r. Thus, $r^2 = (4-r)^2 + (3-r)^2$. Then $r^2 - 14r + 25 = 0$ and $r = 7 \pm \sqrt{24} = 7 \pm 2\sqrt{6}$. However, $7 + 2\sqrt{6} > 8$ and we reject this value. So, the answer is $r = 7 - 2\sqrt{6}$.

10. Alice walks down to the bottom of an escalator that is moving up. She counts 150 steps. Her friend Bob walks up to the top of the escalator and counts 75 steps. Alice's speed of walking (number of steps per unit time) is 3 times Bob's walking speed. How many steps are visible on the escalator at a given time?

* *A grant in support of this activity was received from the Canadian Mathematical Society.*
La Société mathématique du Canada a donné un appui financier à cette activité.

Solution:

Let u be the speed of the escalator and v be the speed of Bob. Then the speed of Alice is $3v$.

Bob moves at the speed $u+v$ up to the top and Alice moves at the speed $3v-u$ down to the bottom of the escalator that moves up.

If x is the number of steps visible on the escalator at a given time, $t_A = \frac{x}{3v-u}$ is the time of Alice and $t_B = \frac{x}{v+u}$ is the time of Bob to cover the distance x.

On the other hand, $t_A = \frac{150}{3v}$ and $t_B = \frac{75}{v}$. Thus,

$$\frac{t_A}{t_B} = \frac{2}{3} = \frac{v+u}{3v-u}, \quad \text{so} \quad v = \frac{5}{3}u.$$

Then, $3v = 5u$ and $3v - u = 4u$. Now, from the fact that

$$t_A = \frac{x}{3v-u} = \frac{150}{3v},$$

we have

$$\frac{x}{4u} = \frac{150}{5u}, \quad \text{so} \quad x = 120.$$

Answer: 120 steps.

* *A grant in support of this activity was received from the Canadian Mathematical Society.*
La Société mathématique du Canada a donné un appui financier à cette activité.