1. Consider the initial value problem (IVP):

\[ y' = f(t, y), \quad y(t_0) = y_0. \]

(a) Write down a set of conditions on the scalar function \( f(t, y) \) so that this IVP has a unique solution in some interval containing \( t_0 \).

(b) Outline the method of successive approximations (i.e., the Picard iteration method) for the existence of a solution to the IVP.

2. Assume that \( \omega^2 \neq 4 \). Use the Laplace transform to solve the initial value problem

\[ y'' + \omega^2 y = \cos 2t, \quad y(0) = 1, \quad y'(0) = 0. \]

3. Let \( C((\alpha, \beta), \mathbb{R}^n) \) be the standard vector space over \( \mathbb{R} \) of all continuous functions from \( (\alpha, \beta) \) to \( \mathbb{R}^n \), and \( P(t) \) be a continuous \( n \times n \) matrix function on \( (\alpha, \beta) \). Define the set

\[ X = \{ x \in C((\alpha, \beta), \mathbb{R}^n) : x(t) \text{ is a solution of } x' = P(t)x \text{ on } (\alpha, \beta) \}. \]

Show that \( X \) is an \( n \)-dimensional vector space over \( \mathbb{R} \).

4. (a) Give the definition of the (Liapunov) stability, instability, and asymptotic stability of an equilibrium point \( x^* \in D \) for the autonomous system \( x' = f(x) \), where \( f \) is a Lipschitz continuous vector field on the domain \( D \subset \mathbb{R}^n \).

(b) For two dimensional system \( u' = v, \quad v' = -u \), prove that the equilibrium point \( (0, 0) \) is stable but not asymptotically stable.
5. Use the method of separation of variables to solve the Laplace equation
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]
inside a rectangle \(0 \leq x \leq L, 0 \leq y \leq H\), with the boundary conditions \( \frac{\partial u}{\partial x}(0, y) = 0, \frac{\partial u}{\partial x}(L, y) = 0, u(x, 0) = 0, \) and \( u(x, H) = f(x) \).

6. (a) Verify that \( v(t, x) = \sum_{n=1}^{\infty} a_n \sin(nx)e^{-(n\pi)^2t} \) is a solution of the heat equation \( \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, t > 0, x \in (0, 1) \), subject to the boundary condition \( v(0, t) = 0 \) and \( v(1, t) = 0 \), and give the formula for \( a_n \) in terms of the initial function \( v(x, 0) \).

(b) Find a solution of the nonhomogeneous problem
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad t > 0, x \in (0, 1), \]
\[ u(0, t) = 20, \quad u(1, t) = 11, \quad t \geq 0, \]
\[ u(x, 0) = f(x), \quad x \in [0, 1]. \]