Note: There are 6 problems and use separate answer booklet for each problem. You have 3 hrs to complete this examination

- **Problem 1:**

  a) Let $X_1, X_2, \ldots, X_n, \ldots$ be independent and identically distributed random variables. Prove that

  \[ E(|X_1|) < \infty \quad \text{if and only if} \quad P\{|X_n| > n \text{ i.o.}\} = 0. \]

  **Note:** For a sequence of random events $A_1, A_2, \ldots, \{A_n \text{ i.o.}\}$ is defined as $\lim_{m \to \infty} \bigcup_{n>m} A_n$

  b) At a party $n$ people put their cell-phones in box where the cell-phones are mixed together. Each person then randomly select one. If we denote the random variable $X =$ the number of people who select their own cell-phone. Find the expected value $E(X)$ and the variance $\text{Var}(X)$.

- **Problem 2:**

  (a) Let $A$ and $B$ are two $p \times p$ symmetric matrices. Also let $f$ be a matrix function defined as (notations are usual)

  \[ f(B) = \log|B| + \text{tr}[B^{-1}A]. \]

  If $A > 0$ , then, subject to $B > 0$, prove that $f(B)$ is minimized uniquely at $B = A$.

  (b) Now suppose that $y_1, \ldots, y_i, \ldots, y_n$ are $n$ independent $p$–dimensional observations taken from $N_p(\mu, \Sigma)$ distribution. Also suppose that $A = \sum_{i=1}^{n}(y_i - \bar{y})(y_i - \bar{y})'$, with $\bar{y} = \sum_{i=1}^{n} y_i/n$, is a $p \times p$ sample sum of products matrix. Use the result from (a) and prove that $A/n$ is the maximum likelihood estimate of $\Sigma$. 
Problem 3:

Consider a standardized linear combination (SLC) of $y$, namely $z_1 = a_1'y$. Here $\sum_{i=1}^{p} a_{1i}^2 = 1$.

a) Then show that no SLC of $y$ has a variance larger than $\lambda_1$, the variance of the first principal component.

b) If $z_{k+1} = a_{k+1}'y$ is a SLC of $y$ which is uncorrelated with the first $k$ principal components of $y$, then show that the variance of $z_{k+1}$ is maximized when $z_{k+1}$ is the (k+1)th principal component of $y$.

c) Suppose that $y \sim N_3(0, \Sigma)$, with

\[
\Sigma = \begin{bmatrix}
\beta^2 + \delta & \beta & \beta \\
\beta & 1 + \delta & 1 \\
\beta & 1 & 1 + \delta
\end{bmatrix}.
\]

Find the eigenvalues of $\Sigma$ and the first principal component of $y$.

Problem 4:

Consider $2n$ independent random variables $X_i, Y_i, i = 1, \ldots, n$, where $X_i \sim N(\mu_i, \sigma^2)$ and $Y_i \sim N(\mu_i, \sigma^2)$. $\mu_i, i = 1, \ldots, n$ and $\sigma^2$ are the unknown parameters.

a) Find the maximum likelihood estimators for all the unknown parameters.

b) Let $S_i = (X_i - Y_i)/2$ and $T_i = (X_i + Y_i)/2$, for $i = 1, \ldots, n$. Are $S_i$ and $T_i$ independent?

c) Find the maximum likelihood estimators for the unknown parameters based on $S_i, T_i, i = 1, \ldots, n$.

d) Compare the results from a) and c). Which method provides better estimator for $\sigma^2$?

Problem 5:

Let $X_1, X_2, \ldots, X_n$ be a random sample with common probability mass function $f(x; \theta) = \frac{\theta^x}{x!}e^{-\theta}$, for $x = 0, 1, 2, \ldots$, zero elsewhere.

a) Determine the mle of $\theta$.

b) Find a sufficient and complete statistic for $\theta$.

c) Determine the minimum variance unbiased estimator of $\theta$. 
Problem 6:

Let $X_{1,1}, X_{1,2}, \cdots, X_{1,n_1}, X_{2,1}, X_{2,2}, \cdots, X_{2,n_2}, \cdots$ and $X_{k,1}, X_{k,2}, \cdots, X_{k,n_k}$ be independent random samples from the normal distributions $N(\mu_i, \sigma^2_i)$ for $i = 1, 2, \cdots, k$.

a) Find the likelihood ratio statistic for the test of hypothesis $H_0 : \sigma^2_1 = \sigma^2_2 = \cdots = \sigma^2_k$ against all alternatives.

b) When $k = 2$, find a statistic with known distribution such that the likelihood ratio statistic in a) is a function of it.