Memorial University of Newfoundland

Department of Mathematics and Statistics

PhD Qualifying Examination – Topology

Name:

Student Id:

There are 8 problems. The duration of the exam is 3 hours. Complete solutions to six problems constitutes a perfect paper.

1. (a) Define what it means for a topological space to be connected, and what it means to be path-connected.
   (b) Show that if $U$ is a connected open subset of $\mathbb{R}^n$ with the standard topology then $U$ is path-connected.
   (c) Show that if $X$ is a connected space and $f : X \to Y$ is continuous and surjective then $Y$ is a connected space.
   (d) Provide an example of a connected space that it is not path-connected. No proof is required.

2. Let $(X, \text{dist}_X)$ and $(Y, \text{dist})$ be metric spaces, and let $f : X \to Y$ be a function. Shows that the following statements are equivalent.
   (a) For every $a \in X$ and every $\epsilon > 0$ there is $\delta > 0$ such that for every $x \in X$, if $\text{dist}_X(a, x) < \delta$ then $\text{dist}(f(a), f(x)) < \epsilon$.
   (b) For every open subset $V$ of $Y$, the preimage $f^{-1}(V)$ is an open subset of $X$.
   (c) For every closed subset $C$ of $Y$, the preimage $f^{-1}(C)$ is a closed subset of $X$.

3. Let $X$ be a topological space.
   (a) For a topological space, define what it means to be regular and what it means to be compact.
   (b) Show that if $X$ is compact and Hausdorff then $X$ is regular.
   (c) Let $\mathbb{R}_l$ be the set of real numbers with the topology having the intervals $[a, b)$ as a basis. Is $\mathbb{R}_l$ compact? Is $\mathbb{R}_l$ regular? Justify your answers.

4. (a) Let $p : X \to Y$ be a surjective map where $X$ is a topological space and $Y$ is a set. Define the quotient topology on $Y$ induced by $p$.
   (b) Let $I = [-1, 1]$ be closed interval as a subspace of $\mathbb{R}$. Identify the points $-1$ and $1$ of $I$ to a single point in order to form a quotient set $Q$. Endow $Q$ with the quotient topology. Give a detailed proof that $Q$ is homeomorphic to the subspace $S^1 = \{x \times y \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ of $\mathbb{R}^2$ with the standard topology.
5. Let $X$ be a topological space.
   (a) Define what it means for a topological space to be second countable.
   (b) Let $X$ be a compact Hausdorff space. Show that $X$ is metrizable if and only if $X$ is second countable.
   (c) Prove that $\mathbb{R}^2$ with the dictionary order topology is metrizable but it is not second countable.

6. Let $X$ be a topological space and let $(Y, \text{dist}_Y)$ be a metric space.
   (a) Define what it means for a sequence of functions $\{f_n : X \to Y | n \in \mathbb{Z}_+\}$ to converge uniformly to a function $f : X \to Y$.
   (b) Let $\{f_n : X \to Y | n \in \mathbb{Z}_+\}$ be a sequence of continuous functions converging uniformly to $f : X \to Y$. Show that $f$ is continuous.
   (c) Let $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n(1-x)^n$. Does the sequence $\{f_n | n \in \mathbb{Z}_+\}$ converges uniformly?

7. (a) Define what it means for a topological space to be locally compact.
   (b) Let $X$ be a locally compact, Hausdorff space. Define the one-point compactification $\hat{X}$ and define the topology on $\hat{X}$ in detail. No proof is required here.
   (c) Give a detailed proof that the one-point compactification of $\mathbb{R}$ is homeomorphic to the subspace $S^1 = \{x \times y \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ of $\mathbb{R}^2$ with the standard topology.

8. Let $\{X_\alpha | \alpha \in J\}$ be a family of topological spaces.
   (a) Define $\prod_{\alpha \in J} X_\alpha$ as a set, and define the product topology on $\prod_{\alpha \in J} X_\alpha$.
   (b) For $\beta \in J$, show that the projection $P_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ is continuous.
   (c) Let $Z$ be a topological space. Show that a function $f : Z \to \prod_{\alpha \in J} X_\alpha$ is continuous if and only if $P_\alpha \circ f$ is a continuous function for each $\alpha \in J$. 