The exam consists of 4 sections. Solutions to at least one and at most two questions in each section must be submitted. The perfect score will be awarded for 6 questions fully solved. Each whole question carries equal credit. More credit will be given for complete solutions than for a proportionate number of parts.

Allotted time: 3 hours.
Part A: Real Analysis

A1. Consider the series
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n. \] (1)
(a) Determine for which values of \( x \in \mathbb{R} \) the series (1) converges and for which \( x \in \mathbb{R} \) it converges absolutely.
(b) Prove that the series (1) converges uniformly on \([0,1]\).
(c) Suppose \( x \in [0,1] \). Let \( S(x) \) denote the sum of the series (1) and let \( S_n(x) \) denote the \( n \)-th partial sum. Prove that if \( x_n \to 1^- \), then \( S_n(x_n) \to S(1) \) as \( n \to \infty \).
(d) Evaluate \( S(1) \).

A2. (a) Let \( f(x) \) be a real-valued function defined on some subset of \( \mathbb{R}^n \). Give definitions of the following:
\begin{itemize}
  \item \( f(x) \text{ is continuous at the point } x^* \in \mathbb{R}^n \);
  \item \( f(x) \text{ is continuous in the domain } D \subset \mathbb{R}^n \);
  \item \( f(x) \text{ is differentiable at the point } x^* \in \mathbb{R}^n \).
\end{itemize}
(b) Suppose \( f(x) \) is differentiable at \( x^* \in \mathbb{R}^n \) and \( f(x^*) = 0 \). Prove that if \( n > 1 \), then
\[ \liminf_{x \to x^*} \frac{|f(x)|}{\|x - x^*\|} = 0. \]
(c) Does the statement (b) hold true in the case \( n = 1 \)? Explain your answer.

A3. (a) Define the improper Riemann integral \( \iint_{\mathbb{R}^2} f(x, y) \, dx \, dy \).
(b) Show that the Riemann integral
\[ \iint_{\mathbb{R}^2} (x^2 + y^2 + 1)^{-s} \, dx \, dy \quad (s \in \mathbb{R}) \]
exists if and only if \( s > 1 \).
Part B: Measure and Integration

B1. (a) Suppose $f(x)$ is Lebesgue integrable on $[0, 1]$. Show that the following statements are equivalent:

(a) $\int_E f = 0$ for each open set $E \subset [0, 1]$;
(b) $\int_E f = 0$ for each measurable set $E \subset [0, 1]$;
(c) $f(x) = 0$ for almost every $x \in [0, 1]$.

B2. (a) State the Monotone Convergence Theorem for nonnegative measurable functions.

(b) Let $N$ be a positive integer. Prove that for any $x \in (0, N)$ the sequence $\{(1 - \frac{x}{n})^n\}$, $n = N, N + 1, N + 2, \ldots$, is increasing. [Suggestion: Use logarithmic differentiation.]

(c) Use (a) and (b) to prove that for any $f \geq 0$ defined on $[0, \infty)$ and such that $f(x)e^{-x}$ is integrable the following holds:

$$\lim_{n \to \infty} \int_0^n \left(1 - \frac{x}{n}\right)^n f(x) \, dx = \int_0^\infty e^{-x} f(x) \, dx.$$  

Suggestion: Consider $g_n(x) = \begin{cases} f(x)(1 - x/n)^n, & 0 < x < n \\ 0, & x > n \end{cases}$.

B3. (b) Suppose $f$ is a bounded function on $[0, 2\pi]$ and $A \subset [0, 2\pi]$ is a measurable set. Show, referring to appropriate facts or constructions of measure theory, that $\forall \varepsilon > 0$ there exists a finite union $U$ of open intervals such that

$$\left|\int_U f - \int_A f\right| < \varepsilon.$$  

(b) Show (by calculation) that for any finite interval $I \subset \mathbb{R}$

$$\lim_{n \to \infty} \int_I \cos nx \, dx = 0.$$  

(c) Let $A \subset [0, 2\pi]$ be a measurable set. Prove that

$$\lim_{n \to \infty} \int_A \cos nx \, dx = 0.$$
Part C: Complex Analysis

C1. (a) Explain why the following function is analytic in some neighborhood of 0 (including the point $z = 0$):

$$f(z) = \begin{cases} \frac{z}{e^z - 1}, & z \neq 0, \\ 0, & z = 0 \end{cases}.$$

(b) Find the radius of convergence of the Maclaurin series for the function $f(z)$ defined in (a).

(c) Find all singular points of the function $f(z)$ and determine their type: essential singularity, pole (of which order?), branch point, etc.

C2. (a) Find all values of the real and imaginary part of the multi-valued function $\ln(x + iy)$ in terms of $x$ and $y$.

(b) Show that the function $\arctan(y/x)$ is harmonic. Assume for simplicity that $x, y > 0$ and consider the principal branch $\arctan(y/x) \in (0, \pi/2)$.

(c) Prove that if a polynomial $p(z)$ has zero of order $n$ at $z = z_0$ and no other zeros in the region $|z - z_0| \leq R$, then

$$\frac{1}{2\pi i} \oint_{|z-z_0|=R} \frac{p'(z)}{p(z)} = n.$$

C3. Prove that for any $u \in (0, \pi/2)$

$$\int_0^{2\pi} \frac{dt}{(1 + \cos u \cos t)^2} = \frac{2\pi}{\sin^3 u}.$$

Hint: The substitution $\cos t = (z + z^{-1})/2$ results in a contour integral of a rational function.
Part D: Functional Analysis

D1. (a) Prove that the space $l^2$ of infinite complex sequences $(x_n)$, $n = 1, 2, \ldots$, with scalar product $\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$ is a Hilbert space. (Show that all requirements of the definition are met).

(b) Show that the set $\{e^{(k)}\}_{k=1,2,\ldots}$ is an orthonormal basis in $l^2$, where $e^{(k)} = (0, 0, \ldots, 1, 0, \ldots)$ (1 at place $k$). Find the distance $\|e^{(j)} - e^{(k)}\|$.

(c) Define the notion of a compact operator in a (separable) Hilbert space. Prove that the identity operator $I$ in $l^2$ is not compact.

D2. (a) Let $X$ and $Y$ be two normed spaces over $\mathbb{R}$, and $T : X \to Y$ a linear operator. State definitions of the following properties/concepts:

- $T$ being a continuous linear operator;
- $X^*$, the dual (conjugate) space to $X$ (Define the vector space operations and the norm on $X^*$);
- the conjugate operator $T^* : Y^* \to X^*$.

(b) Suppose $X$ and $Y$ are finite-dimensional Euclidean spaces of dimensions, respectively, $m$ and $n$, with orthonormal bases, respectively, $\{e_j\}_{j=1,\ldots,m}$ and $\{f_i\}_{i=1,\ldots,n}$. Let $T : X \to Y$ be defined by the matrix $T_{ij}$, so that $T(\sum x_j e_j) = \sum_{i,j} T_{ij} x_j f_i$. Find an explicit formula for $T^*$.

D3. (a) State the definition of a Cauchy sequence in a metric space and the definition of a complete metric space.

(b) Let $B$ be the set of all bounded real sequences $(x_n)$, $n = 1, 2, \ldots$. Prove that the following function is a metric on $B$:

$$\rho(x, y) = \sup_{n \geq 1} \frac{|x_n - y_n|}{n}.$$  

(c) Prove that the metric space $(B, \rho)$ as defined in (b) is not complete.

(d) Give the definition of a Banach space. Introduce a vector space structure and a norm on the set $B$ defined in (b) so as to make $B$ a Banach space. Give an explicit formula for the metric induced by your norm.