This is a **3 hour** examination divided into three parts. You must attempt at least one question from each part. Complete solutions to **FIVE** questions constitutes a perfect paper. All questions have equal weight.

**Notation.**

- $\mathbb{Z}$ the integers
- $\mathbb{Z}_n$ the ring of integers mod $n$
- $\mathbb{Q}$ the rationals
- $\mathbb{R}$ the real numbers
- $\mathbb{C}$ the complex numbers
- $M_n(R)$ the ring of $n \times n$ matrices over a ring $R$
PART A: Groups

1. (a) Prove that $S_4$ contains a subgroup $H$ isomorphic to $D_4$.
   (b) Determine whether or not $H$ is normal in $S_4$.
   (c) Prove that $A_4$ is not simple.

2. (a) If a group $G$ has a subgroup $H$ of finite index $n$, show $G$ has a normal subgroup $N \subseteq H$ of finite index dividing $n!$.
   (b) Suppose $G$ is a group of finite order divisible by a prime $p$, but $|G| \neq p$. Let $n_p$ be the number of Sylow $p$-subgroups of $G$. If $G$ is simple, show that $|G| \mid n_p!$.
   (c) Using part (b) (or other means), prove that a group of order 36 is not simple.

3. (a) Let $G$ be a group of order $p^2q^2$, where $p$ and $q$ are distinct primes. If $p > q$ and $|G| \neq 36$, show that $G$ has a normal $p$-subgroup.
   (b) Enumerate up to isomorphism all abelian groups of order 200. Your list should be complete and contain no repetitions.

PART B: Rings and Modules

4. Let $A$ be an algebra with 1 over a field $k$ which is algebraic over $k$; that is, every $a \in A$ satisfies a polynomial equation with coefficients in $k$.
   (a) If $ab = 1$ with $a, b \in A$, show that $ba = 1$.
   (b) If $a$ is a left zero divisor in $A$, show that $a$ is a right zero divisor.
   (c) Prove that a nonzero element $a \in A$ is a unit if and only if it is not a zero divisor.

5. Let $A$ and $B$ be right modules over a ring $R$ with $A \subseteq B$. We say that $B$ is an essential extension of $A$ if every nonzero submodule of $B$ intersects $A$ nontrivially.
   (a) Show that $Q$ is an essential extension of $Z$, as $Z$-modules.
   (b) Show that $R$ is not an essential extension of $Q$, as $Z$-modules.
   (c) If $N$ is a submodule of an $R$-module $M$, show that $M$ has a submodule $E$ that is maximal with respect to the property $E \cap N = \{0\}$.

6. (a) What is meant by the Jacobson radical of a ring?
   (b) If $R$ is a ring with 1, $u = a + x$ is a unit and $x \in J(R)$, prove that $a$ is a unit.
   (c) Prove that the Jacobson radical of an artinian ring is nilpotent.
PART C: Linear Algebra and Fields

7. Let $T: V \rightarrow W$ be a linear transformation from a vector space $V$ to a vector space $W$.

(a) Define the terms kernel, image, nullity and rank of $T$.

(b) Given that $V$ and $W$ are finite dimensional, state and prove a theorem relating the nullity and rank of $T$.

(c) If $V$ and $W$ have the same finite dimension, show that $T$ is one-to-one if and only if $T$ is onto.

8. Let $\text{tr}: M_n(\mathbb{R}) \rightarrow \mathbb{R}$ denote the trace map.

(a) Prove that $\text{tr} \ AB = \text{tr} \ BA$ for all $n \times n$ matrices $A$ and $B$.

(b) Suppose $S: M_n(\mathbb{R}) \rightarrow \mathbb{R}$ is a linear transformation satisfying $S(AB) = S(BA)$ for all $A, B$ in $M_n(\mathbb{R})$. Show that there exists a real number $k$ such that $S(A) = k \text{tr}(A)$ for all $A$ in $M_n(\mathbb{R})$.

9. (a) Suppose $F$ is a finite field. Prove that there are irreducible polynomials of arbitrarily high degree in the polynomial ring $F[x]$.

(b) Can an algebraically closed field be finite? Explain.

10. Let $F = \{a + b\alpha + c\alpha^2 \mid a, b, c \in \mathbb{Q}\}$ where $\alpha$ is the real cube root of 2.

(a) Prove that $F$ is a field.

(b) Prove that $\sqrt{2} \notin F$. 