Slicing inequalities for measures of convex bodies

Abstract:

We consider the following problem. Does there exist an absolute constant $C$ such that for every $n \in \mathbb{N}$, every integer $1 \leq k < n$, every origin-symmetric convex body $L$ in $\mathbb{R}^n$, and every measure $\mu$ with non-negative even continuous density in $\mathbb{R}^n$,

$$\mu(L) \leq C^k \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) \frac{|L|^{k/n}}{k},$$

(1)

where $\text{Gr}_{n-k}$ is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$, and $|L|$ stands for volume? This question is an extension to arbitrary measures (in place of volume) and to sections of arbitrary codimension $k$ of the hyperplane conjecture of Bourgain, a major open problem in convex geometry.

We show that (1) holds for arbitrary origin-symmetric convex bodies, all $k$ and all $\mu$ with $C \sim \sqrt{n}$, and with an absolute constant $C$ for some special classes of bodies, including unconditional bodies, unit balls of subspaces of $L_p$, and others. We also prove that for every $\lambda \in (0, 1)$ there exists a constant $C = C(\lambda)$ so that inequality (1) holds for every $n \in \mathbb{N}$, every origin-symmetric convex body $L$ in $\mathbb{R}^n$, every measure $\mu$ with continuous density and the codimension of sections $k \geq \lambda n$. The latter result is new even in the case of volume.

The proofs are based on a stability result for generalized intersection bodies and on estimates of the outer volume ratio distance from an arbitrary convex body to the classes of generalized intersection bodies.