1. Solve the system
\[
\begin{align*}
3^{2x-y} &= 27 \\
2^{3x+2y} &= 32
\end{align*}
\]

**Solution:** The given system is equivalent to
\[
\begin{align*}
3^{2x-y} &= 3^3 \\
2^{3x+2y} &= 2^5
\end{align*}
\]
So equating exponents, we must have \(2x - y = 3\) and \(3x + 2y = 5\). Solving this system gives \((x, y) = (11/7, 1/7)\).

2. Prove the identity
\[
\frac{2016^{-x}}{2016^{-x} + 1} + \frac{2016^x}{2016^x + 1} = 1.
\]

**Solution:** We show the left hand side and right hand are equal for all values of \(x\). Algebra gives
\[
\frac{2016^{-x}}{2016^{-x} + 1} + \frac{2016^x}{2016^x + 1} = \frac{2016^{-x}(2016^x + 1) + 2016^x(2016^{-x} + 1)}{(2016^{-x} + 1)(2016^x + 1)}
= \frac{1 + 2016^{-x} + 1 + 2016^x}{1 + 2016^{-x} + 2016^x + 1}
= 1.
\]

3. (a) If \(\log_3(\log_4(a^3)) = 1\), find \(a\).
   (b) Let \(a > 1\). Find all possible solutions for \(x\) such that the following equation holds:
   \[
   \log_a x + \log_a(x - 2a) = 2
   \]

**Solution:**
(a) \(\log_a b = c\) is equivalent to \(a^c = b\). So the equation is equivalent to \(3^1 = \log_4(a^3)\). And this is equivalent to \(4^3 = a^3\), hence \(a = 4\).
(b) The equation is equivalent to
\[
a^{\log_a x + \log_a(x - 2a)} = a^{\log_a x} \cdot a^{\log_a(x - 2a)} = x(x - 2a) = a^2
\]
which leads to the quadratic equation \(x^2 - 2xa - a^2 = 0\) which gives \(x_{\pm} = a \pm \sqrt{2}a\). We cannot have \(x_-\), as this does not solve the original equation (since logarithms of negative numbers are not defined). Thus the only solution is \(x_+ = a(1 + \sqrt{2})\).
4. (a) Prove that
\[ S = 1 + 2 + 3 + \cdots + (n - 1) + n = \frac{n(n + 1)}{2} \]
for any natural number \( n \geq 1 \). (Hint: Write the sum backwards and forwards and add the results!)

**Solution:** See below.

(b) Determine the value of \( n > 0 \) for which
\[ 2^1 \cdot 2^2 \cdot 2^3 \cdots 2^n = 2^{210}. \]

**Solution:** Using the laws of exponents the expression is equivalent to
\[ 2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{210}. \]

So we require \( 1 + 2 + 3 + \cdots + n = 210 \). Now let \( S = 1 + 2 + 3 + \cdots + n \), so \( S \) is also given by \( S = n + (n - 1) + (n - 2) + \cdots + 1 \). Adding these two expressions we have
\[ 2S = (n + 1) + (n + 1) + \cdots + (n + 1) = n(n + 1) \]
or
\[ S = \frac{n(n + 1)}{2}. \]

Hence we need to solve
\[ \frac{n(n + 1)}{2} = 210 \quad \text{or} \quad n^2 + n - 420 = 0. \]

Factoring the quadratic, we have \((n + 21)(n - 20) = 0\), which says \( n = -21 \) or \( n = 20 \). Since we are looking for the positive value of \( n \) then \( n = 20 \).

5. Determine the number of integer values of \( x \) such that \( \sqrt{2 - (1 + x)^2} \) is an integer. Fully justify that you have identified the correct number.

**Solution:** For \( \sqrt{2 - (1 + x)^2} \) to be an integer, then \( 2 - (1 + x)^2 \) must be a perfect square. So we consider \( 2 - (1 + x)^2 = 0, 1, 4, 9, 16, \ldots \). Hence we consider \( (1 + x)^2 = 2, 1, -2, -7, \ldots \). Of course \( (1 + x)^2 \) must be nonnegative, so we only have \( (1 + x)^2 = 2 \) or \( (1 + x)^2 = 1 \). This gives \( x = \pm \sqrt{2} - 1 \) or \( x = \pm 1 - 1 \). Only the later gives integer values of \( x \). So we have the integer solutions \( x = 0 \) or \( x = -2 \). Hence there are two integer solutions.

6. Find all values of \( k \) so that \( x^2 + y^2 = k^2 \) will intersect the circle with equation
\[ (x - 5)^2 + (y + 12)^2 = 49 \]
at exactly one point.

**Solution:** See Figure 1 on the next page. The equation \( x^2 + y^2 = k^2 \) is a circle with centre \((0, 0)\) and radius \( |k| \). The other equation represents a circle with centre \((5, -12)\) and radius 7. To intersect at one point the two circles must share a common tangent at this point and their centres and the point of tangency all fall on a straight line. There are two such circles. The centre \((5, -12)\) is 13 units from the origin. Hence the radius of circle one (that we are trying to find) is then \( 13 - 7 = 6 \) or \( 13 + 7 = 20 \). Hence \( k = 6 \) or \( k = 20 \).
7. When a two digit number and a three digit number are multiplied, the result is $7777$. Find the largest such three-digit number possible.

**Solution:** The two and three digit number must be formed as products of the prime factors of $7777$. And $7777$ can be factored as $7777 = 7 \times 1111 = 7 \times 11 \times 101$. Now to make the largest three digit number, we multiply $7 \times 101$, so $7777 = 11 \times 707$. Hence the largest such three digit number is $707$.

8. (a) Prove the identity

$(\sin x)(1 + 2 \cos 2x) = \sin(3x)$.

You may use the identities

$$\sin(x + y) = \sin x \cos y + \sin y \cos x \quad \cos(x + y) = \cos x \cos y - \sin x \sin y.$$ 

(b) Suppose $n$ is a positive integer. Prove the identity

$$(1 + 2 \cos(2x) + 2 \cos(4x) + \ldots + 2 \cos(2nx))(\sin x) = \sin((2n + 1)x).$$

**Solution:** (a) Use $\sin(3x) = \sin(2x + x) = \sin(2x)\cos(x) + \sin(x)\cos(2x)$. Using the above identity to break up the $2x$ arguments, we get

$$\sin(3x) = (2\sin x \cos x) \cos x + \sin x \cos 2x$$

$$= \sin x(2\cos^2 x + \cos 2x) = \sin x(\cos^2 x + (1 - \sin^2 x) + \cos 2x)$$

$$= \sin x(1 + \cos^2 x - \sin^2 x + \cos 2x) = \sin x(1 + 2 \cos 2x).$$

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(b) To get the general result, assume that for some \( n > 1 \),
\[
\sin x (1 + 2 \cos 2x + 2 \cos 4x + \ldots + 2 \cos(2(n-1)x)) = \sin((2n-1)x)
\]
Then explicitly compute
\[
\sin x (1 + 2 \cos 2x + 2 \cos 4x + \ldots + 2 \cos(2(n-1)x) + 2 \cos(nx)) = \sin((2n-1)x)
\]
Thus \( \sin x \) and \( 2 \cos(nx) \) are additive, so
\[
\sin x = \sin x_1 + \sin((2n-1)x)
\]
Hence the identity holds for \( n \) if it holds for \( n - 1 \). Since by (a) the identity holds for \( n = 1 \), using mathematical induction, it must hold for all positive integers \( n \).

9. Calculate the value of the product
\[
P = \left( 1 + \frac{1}{2} \right) \cdot \left( 1 - \frac{1}{3} \right) \cdot \left( 1 + \frac{1}{3} \right) \cdot \left( 1 - \frac{1}{n} \right) \cdot \ldots \cdot \left( 1 + \frac{1}{n} \right) \cdot \left( 1 - \frac{1}{n} \right)
\]
where \( n \geq 1 \) is a positive integer.

**Solution:** Group the terms with positive relative signs and those with negative relative signs together:
\[
P = \left( \frac{3}{2} \right) \cdot \left( \frac{4}{3} \right) \cdot \ldots \cdot \left( \frac{n+1}{n} \right) \cdot \left( \frac{1}{2} \right) \cdot \left( \frac{2}{3} \right) \cdot \ldots \cdot \left( \frac{n-1}{n} \right)
\]
\[
= \left( \frac{n+1}{2} \right) \cdot \left( \frac{1}{n} \right) = \frac{n+1}{2n}.
\]

10. Define the function \( f(x) \) to be the the largest integer less than or equal to \( x \) for any real \( x \). For example \( f(1) = 1 \), \( f(3/2) = 1 \), \( f(7/2) = 3 \), and \( f(7/3) = 2 \). Let
\[
g(x) = f(x) + f(x/2) + f(x/3) + \ldots + f(x/(x-1)) + f(x/x).
\]
   (a) Calculate \( g(4) - g(3) \) and \( g(7) - g(6) \).
   (b) What is \( g(116) - g(115) \) ?

**Solution**
(a) Direct calculation shows \( g(4) = 8 \) and \( g(3) = 5 \) so \( g(4) - g(3) = 3 \). Similarly \( g(7) = 16 \), \( g(6) = 14 \) so \( g(7) - g(6) = 2 \).

(b) Notice that in the above cases, one deals with differences of the form
\[
f \left( \frac{N}{k} \right) - f \left( \frac{N-1}{k} \right).
\]

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Now suppose $N$ is divisible by $k$, $f(N/k) = m$, say, which implies $f((N - 1)/k) = f(N/k - 1/k) = f(m - 1/k) = m - 1$. So $f(N/k) = f((N - 1)/k) + 1$. Conversely if this last statement is true, then $k$ must divide $N$. To see this, suppose it is not true; then $N/k = m + n/k$ for some $1 \leq n < k$. Thus $f(N/k) = f(m + n/k) = m$ but $f((N - 1)/k) = f(m + (n - 1)/k) = m$ since, of course, $(n - 1)/k$ is not an integer either. So in summary $f(N/k) = f((N - 1)/k) + 1$ if and only if $N$ is divisible by $k$, and otherwise $f(N/k) = f((N - 1)/k)$. So when comparing the difference $g(116) - g(115)$, all the terms will cancel except terms of the form

$$f\left(\frac{116}{k}\right) - f\left(\frac{115}{k}\right)$$

where $k$ divides 116. The divisors of 116 are 1, 2, 4, 29, 58, 116. So there are 6 cases, and each difference contributes 1, so the required difference is 6. Note that this is consistent with (a) (the number 4 has 3 divisors, and 7 has 2 divisors).

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