THE TWENTY-SECOND W.J. BLUNDON MATHEMATICS CONTEST*

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1. An automobile went up a hill at an average speed of 30 km/hr and down the same distance at an average speed of 60 km/hr. What was the average speed for the trip?

Let \( d \) be the distance one way, \( t_1 \) the time going up the hill and \( t_2 \) the time going down. Since \( 30t_1 = d = 60t_2 \), then \( t_1 = 2t_2 \). The required speed is \( s \) where \( s = \frac{2d}{t_1 + t_2} \). Hence,

\[
s = \frac{2d}{t_1 + t_2} = \frac{120t_2}{2t_2 + t_2} = \frac{120}{2 + 1} = 40 \text{ km/hr}.
\]

2. Let \( P \) be a point in the interior of rectangle \( ABCD \). If \( PA = 9 \), \( PB = 4 \) and \( PC = 6 \), find \( PD \).

Since \( PD^2 = c^2 + d^2 \), \( c^2 = 9^2 - a^2 \) and \( d^2 = 6^2 - b^2 \), we have

\[
PD^2 = 9^2 - a^2 + 6^2 - b^2 = 81 + 36 - (a^2 + b^2) = 117 - 16 = 101.
\]

Hence \( PD = \sqrt{101} \).

3. Find the area of the region above the \( x \)-axis and below the graph of \( x^2 + (y + 1)^2 = 2 \).

The graph of the equation \( x^2 + (y + 1)^2 = 2 \) is a circle of radius \( \sqrt{2} \) with centre at \((0, -1)\). The circle intersects the \( x \)-axis at \((\pm 1, 0)\). The area of the required region is clearly a quarter of the circle of radius \( \sqrt{2} \) minus the area of the triangle with base length \( \sqrt{2} \) and height \( \sqrt{2} \). That is, the area

\[
= \frac{1}{4} \pi (\sqrt{2})^2 - \frac{1}{2} (\sqrt{2}) (\sqrt{2}) (\frac{\pi}{2}) = \frac{\pi}{2} - \frac{2}{2}.
\]

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4. A square is inscribed in an equilateral triangle. Find the ratio of the area of the square to the area of the triangle.

Let $x$ be the length of each side of the square. Note that the top triangle is equilateral and all the right triangles are 30-60-90 triangles. Using the values of $\tan 60^\circ$ and $\sin 60^\circ$, the sides of the right triangles are calculated as shown. The base of the equilateral triangle is $x + \frac{2x}{\sqrt{3}}$ and the height is $x + \frac{\sqrt{3}x}{2}$. The required ratio is $\frac{x^2}{\frac{1}{2}(x + \frac{2x}{\sqrt{3}})(x + \frac{\sqrt{3}x}{2})} = \frac{4\sqrt{3}}{(2 + \sqrt{3})^2} = \frac{4\sqrt{3}}{7 + 4\sqrt{3}} = \frac{28\sqrt{3} - 48}{7}$. (Note this number is 0.4974 which is close to 1/2.)

5. Find the number of solutions to the equation $2x + 5y = 2005$ for which both $x$ and $y$ are positive integers.

Note that 5 divides evenly into 2$x$ and hence $x$ must have a factor 5. Let $x = 5t$, then $10t + 5y = 2005$ so that $2t + y = 401$. Since $y = 401 - 2t > 0$, then $t < \frac{401}{2}$, so $t \leq 200$. For each positive $t$ there is a positive solution. Hence there are exactly 200 solutions.

6. For what values of $a$ does the equation $4x^2 + 4ax + a + 6 = 0$ have real solutions?

A quadratic equation has real solutions if and only if the discriminant is nonnegative. That is, there are real solutions for those $a$ for which

$$\Delta = (4a)^2 - 4(4)(a + 6) = 16a^2 - 16a - 96 \geq 0.$$ 

After dividing by 16, we have to solve $a^2 - a - 6 = (a - 3)(a + 2) \geq 0$. Hence $a \geq 3$ or $a \leq -2$.

7. Ace runs with constant speed and Flash runs $x$ times as fast, $x > 1$. Flash gives Ace a head start of $y$ metres, and, at a given signal, they start off in the same direction. Find the distance Flash must run to catch Ace.

Let $d$ be the distance Flash must travel to catch Ace, let $v$ be Ace’s speed, and let $t$ be the time needed to catch up. Then we have two expressions for $d$, namely, $d = vxt$ and $d - y = vt$. Eliminating $v$ we have $d - y = \frac{d}{xt}t = \frac{d}{x}$. Hence $d - \frac{d}{x} = y$ and so $d = \frac{xy}{x - 1}$.

8. Show that $3^n - 2n - 1$ is divisible by 4 for any positive integer $n$.

We take two cases. First choose $n$ to be even. Let $n = 2m$. Then $3^n - 2n - 1 = 3^{2m} - 2(2m) - 1 = 3^{2m} - 1 - 4m = (3^m - 1)(3^m + 1) - 4m$. Clearly $3^m - 1$ and $3^m + 1$ are even so 4 divides their product, and hence divides $3^n - 2n - 1$. For $n$ odd we write $n = 2m + 1$. Then $3^n - 2n - 1 = 3^{2m+1} - 2(2m + 1) - 1 = 3^{2m+1} - 3 - 4m = 3(3^m - 1)(3^m + 1) - 4m$. Clearly 4 divides this last expression since, as before, both $3^m - 1$ and $3^m + 1$ are even.
9. If the polynomial \( P(x) = x^3 - x^2 + x - 2 \) has the three zeros \( a, b \) and \( c \), find \( a^3 + b^3 + c^3 \).

Since \( a, b \) and \( c \) are the roots, then

\[
\begin{align*}
    a^3 - a^2 + a - 2 &= 0, \\
    b^3 - b^2 + b - 2 &= 0, \\
    c^3 - c^2 + c - 2 &= 0.
\end{align*}
\]

Adding, we have \( a^3 + b^3 + c^3 = (a + b + c)^2 - 2(ab + bc + ca) - (a + b + c) + 6 \). Since \( a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) \), then

\[
a^3 + b^3 + c^3 = (a + b + c)^2 - 2(ab + bc + ca) - (a + b + c) + 6.
\]

The right side consists of the so-called “symmetric” functions involving the roots. Since

\[
x^3 - x^2 + x - 2 = (x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc,
\]

then \( a + b + c = 1 \), \( ab + bc + ca = 1 \), so \( a^3 + b^3 + c^3 = 1^2 - 2(1) - 1 + 6 = 4 \).

10. A circle of radius 2 is tangent to both sides of an angle. A circle of radius 3 is tangent to the first circle and both sides of the angle. A third circle is tangent to the second circle and both sides of the angle. Find the radius of the third circle.

Let the radius of the third circle be \( x \) and the length of the shortest distance from the vertex to the first circle be \( a \). Then, by similar triangles,

\[
\frac{a + 2}{x} = \frac{a + 7}{3},
\]

and hence \( a = 8 \). By similar triangles again we have

\[
\frac{a + 10 + x}{x} = \frac{a + 2}{2},
\]

so \( \frac{18 + x}{x} = 5 \). Hence \( x = \frac{9}{2} \).

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