1. Solve the system of equations

\[
a^2 - 3(b^2 + c^2 + d^2) = 7
\]
\[
abcd = 330
\]

where \(a, b, c, d\) are prime numbers. How many different quadruples \((a, b, c, d)\) consisting of 4 prime numbers are there that solve the system?

2. Find the values of \(c\) for which the equation

\[
|x + c| + |x - 6| = 10
\]

has an infinite number of solutions.

3. Suppose a pole \(P_1\) of height 360m is placed on the Signal Hill side of the Narrows and, directly across, on the Fort Amherst side of the Narrows, second pole \(P_2\) of height of 40m is built. (You can assume the bottoms of each pole are at the same height above sea level). A taut wire is placed joining the top of \(P_1\) to the foot of pole \(P_2\). Similarly another taut wire is placed connecting the foot of \(P_1\) to the top of \(P_2\). What is the greatest height of a ship that could sail under the wires?

4. Find the solutions to the quadratic equation \(x^2 - 8x + 13 = 0\). Then evaluate the function \(f(x)\) given by

\[
f(x) = \frac{x^4 - 8x^3 + 14x^2 - 8x + 19}{x^2 - 8x + 15}
\]

at the point \(x = a\) where

\[
a = \sqrt{19 - 8\sqrt{3}}.
\]

_Suggestion:_ Relate \(a\) to a solution of the above quadratic equation.

5. Suppose that \(x^5 - 20qx + 8r\) is divisible by \((x-2)^2\) for real numbers \(q, r\). Determine \(q\) and \(r\).
Problem 3A. Find the area of intersection of two circles of radius 1 and centres at $G(1, 0)$ and $F(0, 1)$.

Solution. Connect the two points of intersection of the circles by the segment DK. The area between the segment and the portion of the circle is a quarter of the circle less the right triangle with legs 1 and 1. This area is $\frac{\pi}{4} - \frac{1}{2}$. The area we are looking for is twice larger, so the answer is $\frac{\pi}{2} - 1$.

Problem 3B. A big circle has centre at J, and 4 small circles (with diameters equal to the radius of the big circle) are drawn inside of it as shown in the picture. Determine the fraction of the area of the big circle not inside of either of the 4 small circles.

Solution. Let the big circle have radius $R$. The area of the big circle is $\pi R^2$. Then each of the small circles has area $\pi \left(\frac{R}{2}\right)^2$ and the 4 of them have area $\pi R^2$ again. To find the area of the region inside the big circle and outside of the 4 small circles we need to subtract the 4 small areas from the big, and add the parts that were subtracting twice. So we get $(2\pi - 4)R^2$. Here we used (generalized) result from part A. Thus, the fraction of the area of the big circle not inside of either of the 4 small circles is $(2\pi - 4)/\pi = 2 - 4/\pi$.

Figure 1: Diagram for Problem 6(a).

6. (a) Find the area of intersection of two circles of radius 1 and centres at $G = (1, 0)$ and $F = (0, 1)$.

(b) A large circle has centre at the point $J$ and 4 small circles (with diameters equal to the radius of the larger circle) are drawn inside of it as shown below. Find the fraction of the area of the larger circle not inside any of the 4 small circles.

Figure 2: Diagram for Problem 6(b).

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7. Consider the sum
\[ S = \frac{5^2 + 3}{5^2 - 1} + \frac{7^2 + 3}{7^2 - 1} + \frac{9^2 + 3}{9^2 - 1} + \cdots + \frac{2017^2 + 3}{2017^2 - 1} \]
(a) How many terms are there in \( S \)?
(b) Calculate \( S \).

8. For how many integer values \( n \) does the function
\[ f(n) = \frac{2^{2017}}{3n + 1} \]
take a positive integer value?

9. (a) Consider the graph of the function \( f(x) = x^2 \) and let \((p, q)\) and \((s, t)\) be two distinct points lying on the curve. Show that the line that passes through these two points has a \( y \)-intercept \( b \) that satisfies \( b = -ps \).
(b) Find all real-valued functions \( f(x) \) that have the property that the line connecting two distinct points on the graph of \( f(x) \) has an \( y \)-intercept given by \(-1\) times the product of the \( x \)-coordinates of each point.

10. (a) Recall that the geometric mean-arithmetic mean inequality states that if \( \{a_1, a_2, a_3 \ldots a_n\} \) is a set of positive real numbers, then
\[ \frac{a_1 + a_2 + \ldots + a_n}{n} \geq [a_1 \cdot a_2 \ldots \cdot a_n]^{1/n} \]
with equality if, and only if \( a_i = a \), i.e. all the \( a_i \) are equal. Prove this for \( n = 2 \).
(b) Consider a triangle with sides of length \( a, b, c \) with a perimeter of 2. Show that
\[ abc + \frac{28}{27} \geq ab + bc + ca \]

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