Four-component Gas/Oil Displacement with Constant Pressure Boundaries

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Abstract

This paper presents the analytical solution of four-component gas/oil displacements under constant pressure boundary conditions. All the previous studies in gas/oil displacement problems have been accomplished under the assumption of constant flux boundaries. In practice however, gas flooding projects are often conducted with constant injection pressure and constant producing well pressure. In this work, a novel generation of Buckley-Leverett’s classic fractional flow theory is applied to solve the problem of four-component gas/oil displacements under constant pressure boundaries.

Conservation of mass in a one-dimensional, dispersion-free medium, for a four-component gas/oil displacement system leads to a set of partial differential equations. The solution of the corresponding initial value problem under constant flux boundary conditions consists of rarefaction waves, shock waves and constant states connecting the injection state to the production state. In incompressible systems with constant pressure boundaries, the total volumetric flux is a function of time and hence, the classical Buckley-Leverett theory is not valid. However, the saturation wave structure obtained from the constant flux boundary condition problem can be used in the solution of the associated problem with constant pressure boundaries by determining the flux analytically as a function of time.

The solution for a four-component gas/oil displacement case study is presented. The determination of time dependent volumetric flux from the solution of the constant flux problem is demonstrated. Results are also obtained using a numerical approach and are compared to the analytical results. This indicates that the analytical solution is indistinguishable from the numerical solution as the number of grid blocks in the numerical method approaches infinity. However, a very fine grid is needed for an acceptable solution.

Key Words: Gas injection, constant flux boundary condition, constant pressure boundary condition.
Introduction

Modeling of gas injection has always been of high interest in the oil industry. This is because in any gas injection process, oil recovery can be substantially enhanced by increasing the injection pressure to reach the minimum miscibility pressure. Miscibility helps increase incremental oil recovery by reducing the capillary pressure to zero allowing for the residual oil to be displaced and recovered theoretically in all the pore space. Miscibility is dictated by the pressure and temperature specific phase behavior of individual hydrocarbon components creating one single phase between the injected gas and initial oil. The increment in oil production is a result of complicated interactions between components in the oil and the injection gas. As a consequence of these interactions, a chromatographic separation of components occurs since the components partition between the oil and gas phases to establish a chemical equilibrium. Different components are then transported by the two phases with different velocities. These velocities are functions of imposed pressure, phase saturations, the physical properties of the phases and their interactions with the rock.

An analytical solution to gas/oil displacement problems is an effective tool to precisely investigate the physical mechanism occurring in the porous media. Conservation of mass in an arbitrary control volume in a one-dimensional, dispersion-free displacement leads to a Riemann problem that may be solved analytically. Welge et al. (1961) presented the analytical solution for displacement of oil by enriched gas in a three-component, two-phase flow system. Their work was an extension of Buckley-Leverett (1941) solution for immiscible displacement of oil by water. Wachmann (1964) extended the theory for alcohol injection. Johansen and Winther (1988) analyzed the global Riemann problem for polymer flooding problems with nonlinear adsorption. Johansen and Winther (1989, 1990) later generalized their solution for two-phase immiscible displacements with an arbitrary number of adsorbing components dissolved in the aqueous phase. The first solutions for four-component systems are reported by Monroe et al. (1990). Johansen et al. (2005) and Wang et al. (2005) analyzed the mathematical structure of four-component gas/oil displacement systems with constant and variable K-values. The first solution for three-phase, three-component systems is presented by Laforce and Johns (2004). Laforce and Johns (2005) furthermore applied their solution technique to the remediation of aquifers contaminated with nonaqueous phase liquids. Laforce et al. (2008) also derived the analytical solution for four-component, three-phase gas/oil/water partially miscible displacements.

However, all the mentioned researches have been conducted under the main assumption of a constant flux boundary condition. Johansen and James (2014) extended the known solution of constant flux Riemann problems to constant pressure boundary situations. They derived general expressions for the volumetric flux before, during and after breakthrough of different shock and rarefaction waves. In this paper, their procedure is applied to model a four-component gas/oil displacement with constant pressure boundaries. The mathematical model of the problem is first formulated. Then, it is demonstrated how the solution of this problem with constant flux boundaries can be used to determine the solution of the associated problem with constant pressure boundaries. An example of four-component gas/oil displacement with constant K-values is discussed. Finally, using the known solution of the problem with constant flux boundaries, the total flux $u_T(t)$ and the break through time of each elementary wave is obtained. The pressure distribution is also determined analytically and a comparison of numerical and analytical approaches is finally presented.

Mathematical Model

Writing the fundamental conservation laws for a system containing four-components in a one-dimensional dispersion-free porous medium leads to the following initial value problem:
\[ \frac{\partial C_i}{\partial t} + u_T \frac{\partial F_i}{\partial x} = 0 \quad ; \quad i = 1, 2, 3, \]
\[ C_i(x, t) = C_i^{inj} \quad ; \quad x \leq 0 \quad ; \quad i = 1, 2, 3, \]
\[ C_i(x, 0) = C_i^{ini} \quad ; \quad x > 0 \quad ; \quad i = 1, 2, 3, \] (1)

where \( C_i \) and \( F_i \) are the overall composition and overall flux of component \( i \) which can be calculated using the following equations,

\[ C_i = c_{i1} S + c_{i2} (1 - S), \quad F_i = c_{i1} f + c_{i2} (1 - f), \quad i = 1, 2, 3, \] (2)

where \( c_{i1} \) and \( c_{i2} \) are the composition of component \( i \) in phase 1 and phase 2 which are assumed to be the vapor and liquid phases, respectively. Only three components are considered since \( \sum_{i=1}^{4} C_i = 1 \). The normalized vapor phase saturation is denoted \( S \) and \( f \) is the fractional flow of the vapor phase. For Corey-type relative permeability models, \( f \) is given by

\[ f = \frac{S^2}{S^2 + \left(1 - S^2\right)^2}, \] (3)

where,

\[ M = \frac{\mu_2}{\mu_1}, \] (4)

where \( \mu \) represents the phase viscosities. Certain assumptions are made in the derivation of the model in (??). The main assumptions are: (1) the flow is one-dimensional, (2) the effects of dispersion and diffusion are negligible as are capillary forces, (3) there is no volume change upon mixing, and (4) phases present at any location are in chemical equilibrium. In order to obtain the analytical solutions, K-values are assumed to be constant. The initial value problem (??) is then converted to an eigenvalue problem, as described in Orr (2007). The resulting eigenvalue problem leads to three eigenvalues which are the wave velocities that overall compositions propagate with. The three associated right eigenvectors are the directions in the compositional space along which composition variations satisfy the mass conservation equation. The eigenvalues associated with the initial value problem (??) are

\[ \lambda_1 = \frac{\partial F_i}{\partial C_i}, \quad \lambda_2 = \frac{F_i + q}{C_i + q}, \quad \lambda_3 = \frac{F_i + h}{C_i + h}. \] (5)

Here \( q \) and \( h \) are functions of the geometry of the tie-lines and mathematical expressions of these two variables can be derived for constant K-value systems (Orr 2007). The right eigenvector associated with \( \lambda_1 \) points in the directions of the tie-lines and the right eigenvectors corresponding to \( \lambda_2 \) and \( \lambda_3 \) are associated with nontie-line paths that cross the tie-lines in the composition space. The elementary wave solution of the initial value problem (??) consists of rarefaction waves, shock waves and constant states connecting the injection state to the initial state. When the continuous variation of overall composition is prohibited by velocity constraint, the shocks come into play to satisfy mass conservation.
Solution of the Riemann Problem with Constant Pressure Boundaries

Johansen and James (2014) found that the elementary wave solution of the initial value problem (\eqref{eq:initial_value_problem}) can be used to determine the solution of associated problem with constant pressure boundaries, i.e.

\begin{equation}
\phi \frac{\partial C_i}{\partial t} + u_T(t) \frac{\partial F_i}{\partial x} = 0 \quad ; \quad i = 1, 2, 3,
\end{equation}

\begin{equation}
C_i(x, t) = C_i^{\text{init}} \quad ; \quad x \leq 0 \quad ; \quad i = 1, 2, 3,
\end{equation}

\begin{equation}
C_i(x, 0) = C_i^{\text{init}} \quad ; \quad x > 0 \quad ; \quad i = 1, 2, 3,
\end{equation}

\begin{equation}
p(0, t) = p_{\text{in}} \quad ; \quad p(L, t) = p_{\text{out}}.
\end{equation}

The solution of the Riemann problems with constant volumetric flux condition consists of a number of rarefaction and shock waves and constant states connecting the adjacent waves. In these systems, the total volumetric flux $u_T$ is constant both in $t$ and $x$. However, in systems with constant pressure boundaries, the total volumetric flux $u_T$ is a function of time. If $V_i = \frac{\phi v_i}{\phi} V_i$ is the solution of the constant flux problem, $V_i(t) = \frac{u_T(t)}{\phi} v_i$ will be the solution of the associated problem with constant pressure boundaries, where $v_i$ is the dimensionless eigenvalue velocity. This is the same in both constant flux and constant pressure boundary problems. Hence the only time dependent parameter is $u_T(t)$. Therefore, the problem reduces to finding the total flux as a function of time, $u_T(t)$. In this paper, the methodology developed by Johansen and James (2014) is employed to calculate $u_T(t)$ for a four-component problem as described by (\eqref{eq:elementary_wave_solution}). The procedure to calculate $u_T(t)$ and the breakthrough time of different elementary waves can be divided into three periods:

**Period 1: the solution before the fastest wave $v_N$ breaks through**

The starting point is to integrate Darcy’s law,

\begin{equation}
\frac{\partial p}{\partial x} \frac{\partial p}{\partial x} = \left[ u_T \right]_{x=0}^{x=L} \Rightarrow \Delta p = u_T \int_0^L \frac{dx}{\lambda_T} \Rightarrow u_T = \frac{\Delta p}{\int_0^L \frac{dx}{\lambda_T}},
\end{equation}

where $p$ is the pressure and $\Delta p$ is the differential pressure imposed between inlet and outlet of the medium. Furthermore, $x$ is the distance from inlet and $L$ is the length of the porous medium. Finally, $\lambda_T$ represents the total mobility. Further elaboration of eq. (\eqref{eq:darcy_law}) results in the following equation:

\begin{equation}
u_T(t) = \frac{\Delta p}{A_N x_N + B_N},
\end{equation}

where $N$ is the total number of elementary waves and $x_N$ is the location of the leading edge of the fastest wave. If $N$ is arranged in order of increasing wave velocities, all the terms with subscript $N$ are valid until the leading wave breaks through. After the breakthrough of the leading wave, the number of the elementary waves reduces by one and there will be another leading wave in the system. Hence, $N$ and all the corresponding terms need to be updated for the new wave structure. It is shown in Johansen and James (2014) that $x_N$ satisfies

\begin{equation}
\frac{dx_N}{dt} = \frac{u_T(t)}{\phi} v_N(S_R) = \frac{\Delta p v_N(S_R)}{\phi(A_N x_N + B_N)},
\end{equation}
where \( v_N \) is the eigenvalue of the leading wave and \( S_R \) is the initial gas phase saturation, which also needs to be updated after the breakthrough of each wave. Eq. (??) is a separable ordinary differential equation with solution given by

\[
A_N x_N^2 + 2B_N x_N = C_N t + c,
\]

where,

\[
C_N = \frac{2\Delta p v_N(S_R)}{\phi},
\]

and \( c \) is an integration constant. It depends on the physical location of the leading wave. The variables \( A_N \) and \( B_N \) can be determined by the following equations,

\[
A_N = \sum_{i=1}^{N} r_i + \frac{1}{v_N(S_R)} \sum_{i=1}^{N} l_i - \frac{1}{\lambda_T(S_R)},
\]

\[
B_N = \frac{L}{\lambda_T(S_R)},
\]

where,

\[
r_i = \frac{v_i(S_{i-1}) - v_{i-1}(S_{i-1})}{v_N(S_R)\lambda_T(S_{i-1})},
\]

and,

\[
l_i = \int_{S_{i-1}}^{S_i} \frac{v'_i}{\lambda_T} dS.
\]

As defined in eq. (??), \( r_i \) generally represents the extension of the constant state between two adjacent waves and \( l_i \) in eq. (??) represents the span of the elementary waves on eigenvalue axis. Each elementary wave \( v_i \) starts from \( S_{i-1} \) and ends at \( S_i \). By substituting \( x_N = L \), eq. (??) can be easily solved to find an explicit expression for the time when \( v_N \) breaks through. Knowing the fact that at \( t = 0 \), the leading wave is at \( x_N = 0 \), leads to \( c = 0 \). Hence,

\[
t_{BT,N} = \frac{A_N L^2 + 2B_N L}{C_N},
\]

where \( t_{BT,N} \) is the time when \( v_N \) breaks through and therefore,

\[
u_T(t) = \frac{\Delta p}{\sqrt{B_N^2 + A_N C_N t}}.
\]

**Period 2: the time after \( v_N \) breaks through and before \( v_{N-1} \) breaks through**

When the leading wave breaks through, three different cases may occur:
Case a: the fastest wave \( v_N \) is a shock wave with a constant state behind separating \( v_N \) from \( v_{N-1} \).

In this situation, after the breakthrough of shock \( v_N \), the wave structure becomes a new system with \( N-1 \) elementary waves, where \( v_{N-1} \) becomes the new leading wave. \( u_T(t) \) is then calculated exactly as period 1 by removing \( v_N \) from the system and putting \( S_R = S_{N-1} \).

Case b: the fastest wave \( v_N \) is a shock directly connected to rarefaction wave \( v_{N-1} \).

In this case, rarefaction wave \( v_{N-1} \) starts to continuously break through just after shock \( v_N \). The calculation of \( u_T \) at this period is the same as case c and is described below.

Case c: the fastest wave is a rarefaction wave.

If a rarefaction wave is the fastest wave in the system, \( S_R \) is no longer a constant value after the breakthrough of the wave. Indeed, for wave \( v_N \) it increases from \( S_N \) to \( S_{N-1} \). Figure 1 schematically shows how a rarefaction wave continuously breaks through.

![Figure 1: Rarefaction wave continuously breaking through](image)

If \( S \) is an arbitrary point on rarefaction wave \( v_N \), then \( t_S \), the time when point \( S \) on \( v_N \) breaks through, and \( u_T(t_S) \), the corresponding flux value, are calculated as follows:

\[
t_S = t_{BT,N} + \frac{\phi \left[ L^2 - x(S,t_{BT,N})^2 \right]}{2 \Delta p v_N(S)^2} \left\{ \sum_{i=1}^{N} r_i v_N(S) + \sum_{i=1}^{N-1} l_i + \int_{S_{N-1}}^{S} v_N' \frac{dS}{\lambda T} \right\}, \tag{18}
\]

\[
u_T(t_S) = \frac{\phi \left[ L^2 - x(S,t_{BT,N})^2 \right]}{2 \Delta v_N(S)(t_S - t_{BT,N})}. \tag{19}
\]

In eq. (18), \( x(S,t_{BT,N}) \) is the location of point \( S \) at \( t = t_{BT,N} \) and it is determined using the following equation,

\[
x(S,t_{BT,N}) = \frac{v_N(S)}{\phi} \psi(t_{BT,N}), \tag{20}
\]

where

\[
\psi(t) = \int_0^t u_T(t) dt. \tag{21}
\]
In order to perform the calculations more accurately in this step, we break the entire rarefaction wave into discrete saturation values starting from $S(0) = S_N$ to $S(n) = S_{N-1}$ as follows:

$$
S(0) = S_N,
\vdots
S(n) = S(n-1) + \Delta S = S_{N-1}.
$$

(22)

Since $t_{BT,N}$ and $u_T(t_{BT,N})$ are known, eq. (??) and (??) can be applied to find the $t_{BT,S(1)}$ and $u_T(t_{S(1)})$. These values can be used to calculate the breakthrough time and corresponding flux of $S(2)$ by replacing $t_{BT,N}$ with $t_{BT,S(1)}$ and $S$ with $S(2)$. The procedure is repeated until the whole rarefaction breaks through and corresponding total flux for all discrete saturation values are known.

**Period 3: the time after $v_{N-1}$ breaks through**

When the wave $v_N$ has entirely broken through, the calculations go back to period 1 simply by removing $v_N$ from the system and putting $S_R = S_{N-1}$. Calculations are repeated until all the elementary waves break through.

**Example**

Following the procedure described in previous section, the total volumetric flux and the break through time of different elementary waves are calculated for a four-component gas/oil displacement with constant K-values.

**Elementary wave solution**

In order to derive the solution of the case with constant pressure boundaries, the solution of the problem with constant volumetric flux boundaries must be known. The procedure developed by Johansen et al. (2005) and Wang et al. (2005) is applied to find the solution for this case. Table 1 shows the input parameters for the problem.

<table>
<thead>
<tr>
<th>Component</th>
<th>Composition (mole fraction)</th>
<th>CH₄</th>
<th>CO₂</th>
<th>C₄</th>
<th>C₁₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Oil</td>
<td></td>
<td>0.10</td>
<td>0.20</td>
<td>0.30</td>
<td>0.40</td>
</tr>
<tr>
<td>Injection Gas</td>
<td></td>
<td>0.90</td>
<td>0.10</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>K-values</td>
<td></td>
<td>2.50</td>
<td>1.50</td>
<td>0.50</td>
<td>0.05</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Viscosity Ratio, $M$</td>
<td>10</td>
</tr>
<tr>
<td>Displacement Length, $L$</td>
<td>1 m</td>
</tr>
<tr>
<td>Porosity, $\phi$</td>
<td>0.18</td>
</tr>
<tr>
<td>Inlet Pressure, $p_{in}$</td>
<td>10.0000 MPa</td>
</tr>
<tr>
<td>Outlet Pressure, $p_{out}$</td>
<td>9.6553 MPa</td>
</tr>
<tr>
<td>Differential Pressure, $\Delta p$</td>
<td>0.3447 MPa</td>
</tr>
<tr>
<td>Residual Oil Saturation, $S_{or}$</td>
<td>0</td>
</tr>
<tr>
<td>Connate Water Saturation, $S_{or}$</td>
<td>0</td>
</tr>
<tr>
<td>Critical Gas Saturation, $S_{or}$</td>
<td>0</td>
</tr>
</tbody>
</table>

In the four-component systems, three tie-lines exist: the initial oil tie-line, the injection gas tie-line and one crossover tie-line. The structure of the rarefaction and shock waves constituting the solution depends on the geometry of the tie-lines and the geometry of the tie-lines is reflecting the phase behavior.
of the system through the K-values. The geometry of the tie-lines along with composition paths and the elementary wave solution are all shown in figures 2 and 3. The composition profile for different components is given in figure 4. The numerical solutions using different number of grid blocks are also presented to illustrate how difficult it is to obtain sufficient resolution of the elementary waves. As shown in the figures, increasing the number of grid blocks will result in a solution that is closer to the analytical approach. However, a small degree of numerical dispersion is inevitable even with thousands of grid blocks. This numerical dispersion smears the solution and leads to inaccurate estimation of break through time. Furthermore, figure 2 shows that the composition path in quaternary phase diagram is less sensitive to the number of grid blocks. Since the numerical solutions for 500 and 5000 grid blocks are indistinguishable from analytical solution, only the analytical solution and the numerical solution for the case with 50 grid blocks are shown in the figure.

Figure 2 Composition path for four-component gas/oil displacement

Figure 3 Elementary wave solution for four-component gas/oil displacement

Figure 3 indicates that the solution consists of two rarefaction waves and three shock waves connecting the injection state to the initial state. Starting from initial composition, the solution jumps into the two-phase region by the shock $v_5$. This shock is followed by the rarefaction $v_4$ along the initial tie-line. The initial tie-line is connected to crossover tie-line by the other rarefaction $v_3$. The trailing waves are shocks
Figure 4 Composition profile for different components

\( v_2 \) and \( v_1 \). The shock \( v_2 \) connects the crossover tie-line to injection tie-line and the shock \( v_1 \) takes the solution from two-phase region to injection gas composition point. These elementary waves start and end in key saturation points. Displacement details of this case are provided in table 2.

**Constant pressure boundary solution**

Calculations of the break through time and the total flux before the fastest wave \( v_5 \) breaks through are shown here as a sample. Using the data provided in table 2, \( r_i \) and \( l_i \) are first calculated as follows:

\[
\begin{align*}
  r_1 &= \frac{v_1(S_0) - v_0(S_0)}{v_5(S_R)\lambda_T(S_0)} = 1.3908 \times 10^7, \\
  r_2 &= \frac{v_2(S_1) - v_1(S_1)}{v_5(S_R)\lambda_T(S_1)} = 5.4204 \times 10^7, \\
  r_3 &= \frac{v_3(S_2) - v_2(S_2)}{v_5(S_R)\lambda_T(S_2)} = 7.7571 \times 10^7, \\
  r_4 &= 0, \\
  r_5 &= 0,
\end{align*}
\]

and,

\[
\sum_{i=1}^{N} r_i = 1.4568 \times 10^8. \quad (23)
\]
### Table 2 Key saturation points and eigenvalues

<table>
<thead>
<tr>
<th>Segment</th>
<th>Key saturation point</th>
<th>Saturation value</th>
<th>Eigenvalue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Injection Gas</td>
<td>$S_L = S_0$</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>Trailing</td>
<td>$S_0$</td>
<td>1.0000</td>
<td>0.2194</td>
</tr>
<tr>
<td>Shock $v_1$</td>
<td>$S_1$</td>
<td>0.7744</td>
<td>0.2194</td>
</tr>
<tr>
<td>Constant State</td>
<td>$S_1$</td>
<td>0.7744</td>
<td>0.7367</td>
</tr>
<tr>
<td>Intermediate</td>
<td>$S_1$</td>
<td>0.7744</td>
<td>0.7367</td>
</tr>
<tr>
<td>Shock $v_2$</td>
<td>$S_2$</td>
<td>0.5689</td>
<td>0.7367</td>
</tr>
<tr>
<td>Constant State</td>
<td>$S_2$</td>
<td>0.5689</td>
<td>1.1555</td>
</tr>
<tr>
<td>Intermediate</td>
<td>$S_2$</td>
<td>0.5689</td>
<td>1.1555</td>
</tr>
<tr>
<td>Rarefaction $v_3$</td>
<td>$S_3$</td>
<td>0.4034</td>
<td>1.2237</td>
</tr>
<tr>
<td>Intermediate</td>
<td>$S_3$</td>
<td>0.4034</td>
<td>1.2237</td>
</tr>
<tr>
<td>Rarefaction $v_4$</td>
<td>$S_4$</td>
<td>0.3607</td>
<td>1.5777</td>
</tr>
<tr>
<td>Leading Shock $v_5$</td>
<td>$S_4$</td>
<td>0.3607</td>
<td>1.5777</td>
</tr>
<tr>
<td>Initial Oil</td>
<td>$S_5 = S_R$</td>
<td>0.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

We also have:

\[
l_1 = 0,
\]

\[
l_2 = 0,
\]

\[
l_3 = \int_{S_2}^{S_3} \frac{v_1}{\lambda_T} dS = 2.4791 \times 10^7, \quad \Rightarrow \quad \sum_{i=1}^{N} l_i = 2.1769 \times 10^8. \tag{24}
\]

\[
l_4 = \int_{S_3}^{S_4} \frac{v_4}{\lambda_T} dS = 1.9290 \times 10^8,
\]

\[
l_5 = 0,
\]

we also have:

\[
v_5(S_R) = 1.5777, \tag{25}
\]

\[
\lambda_T(S_R) = 10^{-9},
\]

Hence, $A_N$, $B_N$ and $C_N$ are calculated as follows:

\[
A_N = -7.1634 \times 10^8,
\]

\[
B_N = 10^9, \tag{26}
\]

\[
C_N = 6.0433 \times 10^6,
\]

This leads to the following results for $t_{BT.5}$ and $u_T$. 
\[ t_{\text{BT,5}} = 212 \text{s}, \]
\[ u_T(t) = \frac{344738}{\sqrt{10^{18} - 43.2906 \times 10^{14} t}}; \quad t \leq 212 \text{s}. \]  

**Total velocity**

Figure 5 shows the total velocity as a function of time until all the elementary waves break through. \( t = t_3 \) in this figure is the time when the rarefaction \( v_3 \) is completely broken through. As can be seen, due to different type of elementary waves, \( u_T \) behaves differently at each time period. However, \( u_T \) is generally increasing by the time as the less viscos vapor phase saturation is increasing in the medium.

**Saturation profile**

Once \( u_T \) is known, the location of each saturation can now be calculated at any point in time. Figure 6 indicates the saturation profile at three times before the fastest wave break through: \( t = 0.3 t_{\text{BT,5}} \), \( t = 0.6 t_{\text{BT,5}} \) and \( t = 0.9 t_{\text{BT,5}} \). It can be seen that, the saturation profile is expanding as the total flux \( u_T \) is increasing with time.

**Pressure distribution**

Since \( u_T(t) \) is known, fundamental Darcy’s law can be employed to determine the pressure distribution along the porous media at any certain time,

\[ u_T = -\lambda_T \frac{\partial p}{\partial x} \]  
\[ p_{\text{in}} - p_x = \int_0^x \frac{dx}{\lambda_T}, \]  

where \( p_{\text{in}} \) is the pressure at the inlet and \( p_x \) the pressure value at the distance \( x \) from inlet. For the sake of simplicity, we first determine the pressure distribution along the constant state segment behind the shock.
Figure 6 Vapor phase saturation before the breakthrough of the fastest wave in constant pressure boundary problem

\( v_1 \). The pressure value just behind the shock can be assumed as the inlet pressure for the second constant state piece of the medium. This procedure can be repeated until we reach to a rarefaction wave. Along rarefaction waves, \( \lambda_T \) is no longer a constant value. Indeed, it is a function of vapor phase saturation and a change of variables from \( x \) to \( S \) is needed to perform the integration. Figure 7 shows the pressure distribution along with vapor phase saturation at \( t = 0.6t_{BT5} \).

Figure 7 Pressure distribution along porous medium

**Numerical solution**

The numerical results with different number of grid blocks along with analytical solution is shown in figure 8. Finite difference scheme is applied to solve the initial value problem \( ? \). As shown in the figure, the numerical result is converging to the analytical solution as the number of grid blocks is increasing. However, the numerical dispersion leads to significant errors in the calculation of breakthrough time specially when less number of grid blocks is considered.

**Conclusions**

A generalized fractional flow theory has been successfully applied to a four-component gas/oil displacement. Break through times of different waves and total flux at different times are obtained analytically.
Figure 8 Comparison of analytical and numerical approaches

and numerically. The numerical solution converges slowly to the analytical solution as the number of grid blocks is increasing. This shows that accurate numerical approximations are very time consuming compared to the exact analytical solution.

References