Lecture 1. Van Kampen diagrams

Monday, March 04, 2013
3:35 PM

$G = \langle X \rangle$; Γ - graph $\qquad g \xrightarrow{x} gx$

$V = G$; $E = (g, gx)$, $x \in X$, $g \in G$.

Γ is a geom. object

$g \to gx_1 \to gx_1x_2 \to \cdots \to g\underline{x_1\cdots x_n} = g$

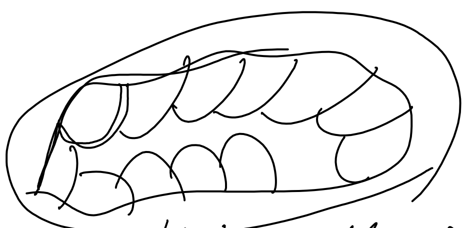
$x_1 \cdots x_n = 1$

$\langle a, b \mid ab = ba \rangle = \mathbb{Z}^2$

$aba^{-1}b^{-1}$ - loop in $\mathbb{Z}^2$

$G = \langle a, b \mid \underline{w_1 = 1, w_2 = 1, \dots, w_k = 1} \rangle$

Suppose that adding discs to Γ corr. to all these words we get a simply connected space.



That means that all relations $w = 1$ in G <u>follow</u> from $w_1 = 1, \dots, w_k = 1$

Logic $\Leftrightarrow$ gr. theory $\Leftrightarrow$ geometry

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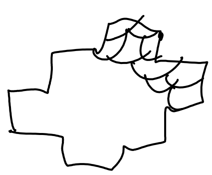


$w = 1$ follows from $w_1 = 1, \dots, w_k = 1$

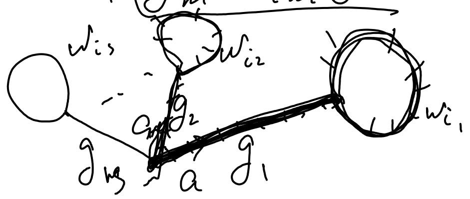
$F_n / \langle\langle w_1, \dots, w_k \rangle\rangle$

normal subgroup gen. by $w_1, \dots, w_k$

$w=1$ in the quotient group $\Longleftrightarrow$

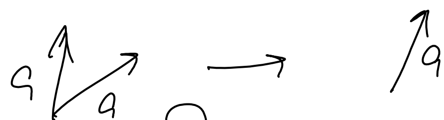
$w \in \langle\langle w_1,\dots,w_k \rangle\rangle = \{ \prod g_m w_{i_m} g_m^{-1}$

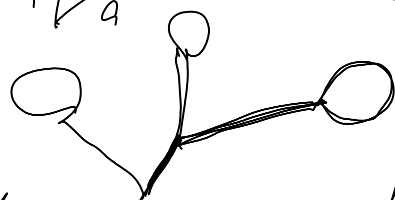
$w = \prod g_m w_{i_m} g_m^{-1}$ in the free group



$a a^{-1} = 1$ in F_n

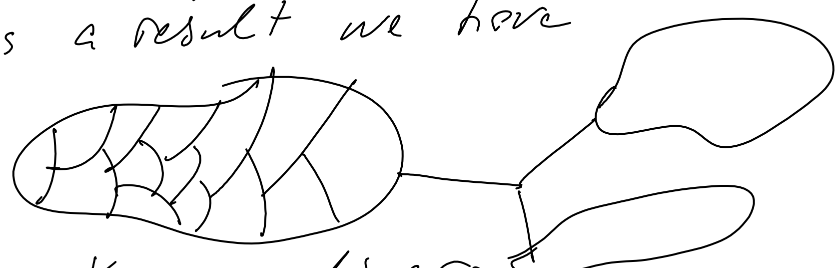
folding





fold as much as we can. Planarity does not change

As a result we have

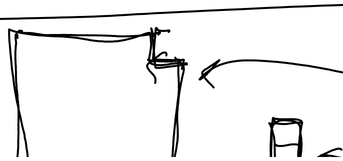


Van Kampen diagram:

Def vK diagram for words is a planar labeled graph with boundary label W, and each cell labeled by w_i

If there is a diagram with ∂ label w, and tile labels w_i, then $w=1 \Leftarrow w_1=1,\dots,w_k=1$.

No way to fill this with dominos



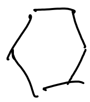
$$a^7 b^7 a^{-1} b\, a^{-7} b^{-7} a\, b^{-1} = 1 \Leftarrow \begin{cases} a b^2 a^{-1} b^{-2} = 1 \\ a^2 b a^{-2} b^{-1} = 1 \end{cases}$$

both hold in S_3

$S_3 = \langle (1,2), (2\,3) \rangle$

$(ab)(ab)(ab)(ab) \neq 1$

$a = (1\,2\,3)$

 - Thurston case

Application to groups

$G > A, B$, $\varphi : A \twoheadrightarrow B$ isomorphism

Can you embed $G \hookrightarrow Q$, such that $\exists g \in Q$

$gAg^{-1} = B$, $gag^{-1} = \varphi(a)$ $\forall a \in A$

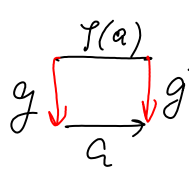
Q is an HNN extension of G with assoc subgroups A, B

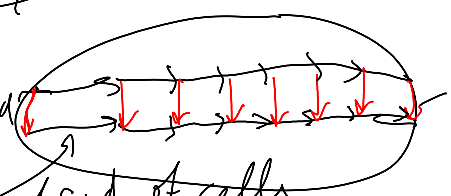
$Q = \langle G, g \mid$ all relations from G and $gag^{-1} = \varphi(a)$ $\forall a \in A \rangle$.

Theorem $G \hookrightarrow Q$.

By contradiction suppose some $u \neq 1$ in G

$u = 1$ in Q



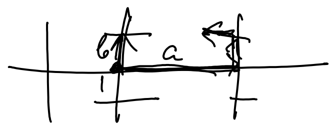


Gromov's polynomial growth thm

$G = \langle X \rangle$, Γ - Cayley graph

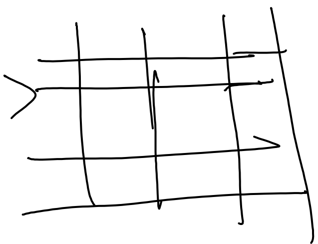
$g \in G$, g — gx $x \in X$

$\langle a, b \rangle$ - free



- free, C. g. of the free group

$\langle a, b \mid ab = ba \rangle$



C. g. of $\mathbb{Z}^2$

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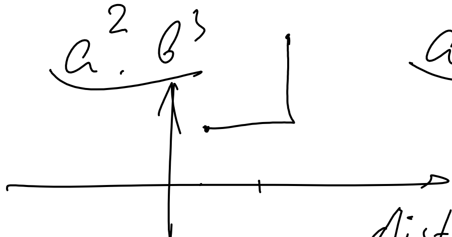
dist(g, h) = length of the shortest path connecting g, h

$= |g^{-1}h|$ - length of the shortest word w: $w = g^{-1}h$ in G

<u>For example</u> $\mathbb{Z}^2 = \langle a, b \rangle$

$\underline{a^2 \cdot b^3}$ $\underline{a^4 \cdot b^5}$



$(a^{-2} b^{-3})\, a^4 b^5 = a^2 b^2$

dist $(a^2 b^3, a^4 b^5) = 4$

Γ

$\forall h \in G$ $h \circ G$ | $h \circ g = hg$

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g — gx

hg — hgx

dist (g, g') = dist (hg, hg')

The mult. by h on the left preserves

The mult. by h on the left preserves the distances $\Rightarrow$ is an isometry

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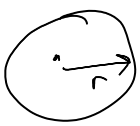
$$g \qquad g'$$

$$\underbrace{g' g^{-1}} \circ g = g' \qquad \Big| \Rightarrow \text{action is transitive}$$

Cayley graph is a "geometry".

Growth

Manifolds; M - manifold



r $V(r)$ - volume of a ball of radius r

Plane: $V(r) \sim r^2$ $\quad |\mathbb{R}^3 \; V(r) \sim r^3$

Hyp-plane $V(r) = \exp(r)$ $\quad |\mathbb{R}^n; V(r) \sim r^n$

Let Γ be a C.g. of $G = \langle X \rangle$

Let $x_0 = e$

$$B_{x_0}(r) = \{x \mid dist(x, x_0) \leq r\}$$

$$V(r) = |B_{x_0}(r)|$$

Isometries take balls to balls

So $|B_x(r)| = |B_{x_0}(r)|$

Švarc $\quad G = \pi_1(M)$

compact

$\pi_1(\bigcirc) = \mathbb{Z}$ $\qquad$ $\pi_1(\bigcirc \times \bigcirc) = \mathbb{Z}^2$

$V_G(r) \sim V_{\tilde{M}}(r)$

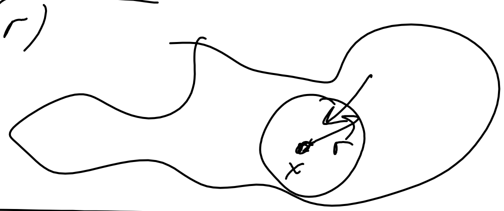
1) Wolff $\quad G$ - nilpotent

$$1 < Z(G) < Z_2(G) < \dots < Z_k(G) = G$$

$GL_n(\mathbb{C}) > UT_n(\mathbb{C})$ - nilpotent group

$V(r)$ is a polynomial

Bass + Guivarc'h found exact polynomial bounding $V(r)$



$|B(r)| \le r^k$

2) $G > H$; $|G:H| < \infty$, then

$$V_G(r) \sim V_H(r)$$

3) G is solvable

$$G > G' > G'' > \dots > 1$$

$V_G(r)$ is polynomial $\Leftrightarrow$ G has a nilpotent subgroup of finite index

Theorem (Gromov) If $V_G(r) < r^k$ then G has a nilp. subgroups of finite index

r^k $(2r)^k = 2^k r^k$	$B(2r)$ is covered by C ($C = 2^k$) of balls of radius r.

$(\Gamma, dist)$; $\Gamma_2 = (\Gamma, \frac{dist}{2}), \dots \Gamma_n = (\Gamma, \frac{dist}{n})$

The limit is some space Γ_∞.

How to define it?

$\Gamma_\infty = \{(g_i), g_i \in G\}$

$dist((g_i),(h_i)) = \left(\frac{dist(g_i, h_i)}{i}\right)$

$\lim(b_i) = b$ if $\forall \varepsilon$ almost all b_i are within ε from b

Ultrafilter $= \{U_j \subseteq \mathbb{N} \mid j \in A\} = \omega$

U_i is "large" 0) finite set are small

a) Every subset of $\mathbb{N}$ is either large or small

b) $V \supset V'$ - large $\Rightarrow$ V is large

c) $V_1 \cap V_2$ - large if V_1, V_2 are large

Theorem There are ultrafilters ~ AC

$\lim^\omega(b_i) = b$ such that for every ε almost all b_i are within ε from b

$\{i \mid b_i \dots\} \in \omega$.

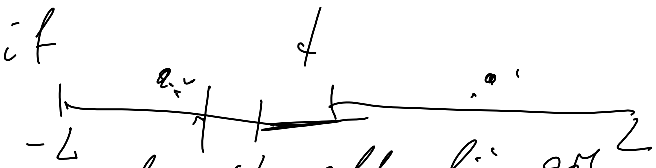
$\lim\{1,2,1,2,1,2,\dots\}$

Thm Every sequence of numbers has a limit

$b_1, b_2, \dots, b_i$ unbounded $\Rightarrow \lim = \infty$

if   +

if



almost all b_i are in one half of the interval

decreasing sequence of intervals, intersection is $\{b\}$.

Lecture 3 | Asymptotic cones

Ultrafilter $\omega = \{U \subseteq \mathbb{N} \mid \quad \}$

Given ω, we can define

$$\lim^{\omega} b_i = b$$

$\forall \varepsilon \quad \{i \mid |b_i - b| < \varepsilon\} \in \omega$

Any sequence of numbers has a limit

Let $(X, dist)$ be a metric space

$$X_i = \left(X, \frac{dist}{i}\right) \quad ; \quad o_i \in X_i$$

$$\overline{\lim}\, X_i = \{(x_i) \mid x_i \in X_i\}$$

$$dist((x_i),(y_i)) = \lim^{\omega} \frac{dist(x_i, y_i)}{i}$$

$$\underline{\widehat{\lim}\, X_i} = \{(x_i) \mid dist((x_i),(o_i)) < \infty\}$$

$(x_i) \sim (y_i)$ if $dist((x_i),(y_i)) = 0$

$\widehat{\lim}/\sim \; - \; Con^{\omega}(X_i, (o_i))$

X – Cayley graph of $G = \langle X \rangle$.

X – homogeneous

Then $Con^{\omega}(X)$ is homogeneous

$(g_i) \circ (x_i) = (g_i x_i)$

$g_i \in \prod_b G \curvearrowright Con^\omega(X)$ – homogeneous

$(y_i x_i^{-1}) \circ (x_i) \longrightarrow (y_i)$

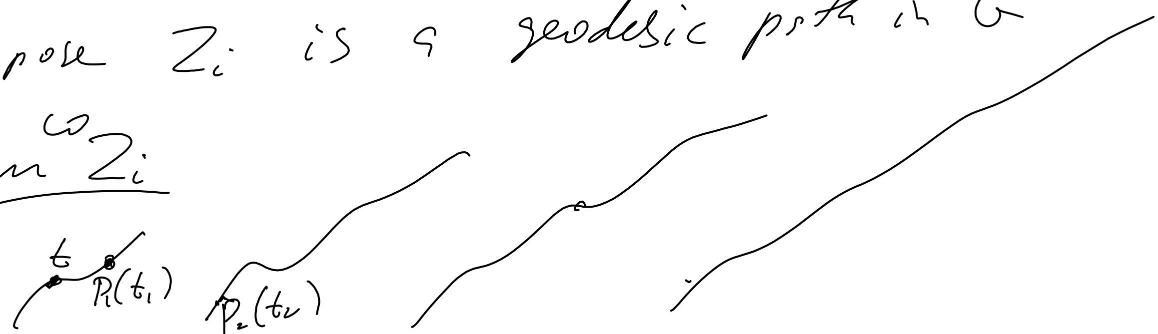
$Con^\omega(G, e) = Con^\omega(G)$

$Z_i \subseteq G$

$\lim Z_i = \{ (z_i) \mid z_i \in Z_i \} \subseteq Con^\omega(G)$

Suppose Z_i is a geodesic path in G

$\lim^\omega Z_i$



$(p_1(t_1), p_2(t_2), \dots) = (p_1(t), p_2(t), \dots)$

$\lim^\omega (t_i) = t$

For every two points in $Con^\omega(G)$ there is a geodesic connecting these points

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Gromov's theorem

G has polynomial growth

$B_G(2r) = \bigcup_{i=1}^{K} B(r) \Rightarrow$

$B_{Con}(2r) = \bigcup_{i=1}^{K} B_{Con}(r)$

$Con^\omega(G)$ – locally compact

H. dim $< \infty$

covering dimension is finite

Montgomery + Zippin $= \Pi_\omega G/\sim$ - Lie group

G
$g \to (g) \in \underline{\Pi_\omega G/\sim}$

$\gamma_i : g \to (h_i^{-1} g h_i)$

$|\gamma_i(G)|$ arbitrary large

$\gamma_i : G \to SL_n(\mathbb{C})$

$G \to \underline{\Pi SL_n(\mathbb{C})/\omega} \quad g \to (\gamma_i(g))$

$\qquad \| $

$SL_n(\Pi \mathbb{C}/\omega) = SL_n(C)$

$\gamma = (\gamma_i)$

$\gamma(G) \leq SL_n(\mathbb{C})$ is infinite

Tits alternative $\Rightarrow \gamma(G)$ either is v. solvable or $\gamma(G)$ contains F_2

Wolf $\Rightarrow \gamma(G)$ is virtually nilpotent

$\qquad \vee$

$\qquad H$ - nilpotent, finite index

$G >_{f.i.} G_0 \twoheadrightarrow H$ - nilpotent

$\qquad \twoheadrightarrow \mathbb{Z}$

$G_0 \xrightarrow{\varphi} \mathbb{Z} \quad | \quad \mathrm{Ker}(\varphi) = N \trianglelefteq G_0$

and if the growth rate of G was n^d
then the growth rate of N is n^{d-1}
use Wolf again

Pansu: $\mathrm{Con}^\omega(G) =$ Lie group with Carnot-Caratheodori nilp

Other applications

$f(x_1, \dots, x_n) = 0$ — f is a polynomial with integer coeff

$x^n + y^n = z^n$

$\underbrace{\mathbb{Z}[x_1, \dots, x_n] / (f)} \xrightarrow{\varphi_i} \mathbb{Z}$

$\varphi_i \Leftrightarrow$ solutions of the equation $f = 0$

inf. many solutions : inf. many hom.

G

$\Gamma \xrightarrow{\varphi_i} G.$ | φ_i, $g^{-1} \varphi_i g$

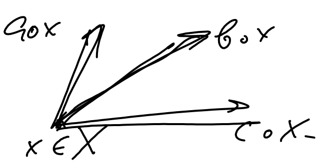
Γ - f.g. , G is f.g.

Let X be the Cayley graph of G

$\gamma \circ_i g = \varphi_i(\gamma) g$ - action by isometries

d_i is the displacement number of $\circ_i$

$\Gamma = \langle a, b, c \rangle$



$d_i \longrightarrow \infty$

$X_1 = X, \quad X_2 = X / d_2, \quad X_3 = X / d_3$

$\underline{\lim} X_i = Con^\omega(G)$

$\circ_i \rightarrow (\circ_i)$ of $\Gamma \curvearrowright Con^\omega(G)$

$\Gamma \curvearrowright Con^\omega(G)$ non-trivially

If G is hyperbolic, $Con^\omega(G)$ is a tree

$\Gamma \rightrightarrows G \longrightarrow$ $\Gamma \curvearrowright$ tree $\Rightarrow$ Γ splits $\Rightarrow$

Drutu + Kapovich "Lectures in geom. group theory"