Identities of algebras and graded algebras with involution.

Irina Sviridova*
Universidade de Brasília, Brazil
Bonne Bay Marine Station, 17/08/2012.

*Supported by CNPq, CAPES and by CNPq-FAPDF PRONEX
grant 2009/00091-0 (193.000.580/2009).

F is a field, $\quad \mathrm{char} F = 0$,
A is an associative $F-$ algebra.

Involution on A is an anti-automorphism of the order 2;
$* : A \to A$
$(\alpha a + \beta b)^* = \alpha a^* + \beta b^*, \qquad (a \cdot b)^* = b^* \cdot a^*,$
$(a^*)^* = a; \qquad a, b \in A, \ \ \alpha, \beta \in F.$

Examples:

1. Complex field $\mathbb{C}$ with conjugation
$x^* = \bar{x} = \alpha - \beta i, \qquad x = \alpha + \beta i \in \mathbb{C}, \qquad \alpha, \beta \in \mathbb{R}$

2. Matrix algebra over F with transposition
$\mathcal{X}^* = \mathcal{X}^t, \ \ \mathcal{X} \in M_n(F).$

3. Matrix algebra $2k \times 2k$ over F with simplectic involution
$$\begin{pmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ \mathcal{X}_3 & \mathcal{X}_4 \end{pmatrix}^s = \begin{pmatrix} \mathcal{X}_4^t & -\mathcal{X}_2^t \\ -\mathcal{X}_3^t & \mathcal{X}_1^t \end{pmatrix}, \ \mathcal{X}_i \text{ are matrices } k \times k.$$

4. $M_n(F) \times M_n(F)^{op}$ with exchange involution
$(X, Y)^* = (Y, X), \qquad X \in M_n(F), \ Y \in M_n(F)^{op}.$
the product of the oposite algebra $A^{op} : a \bullet b := b \cdot_A a.$

2, 3, 4 are all $*$-simple finite dimensional algebras.

Gradings.

G is a group.

$$A \text{ is } G-\text{graded if} \qquad A=\bigoplus_{\theta\in G} A_\theta, \quad A_\theta \subseteq A,$$
$$A_\theta A_\xi \subseteq A_{\theta\xi} \quad \forall \theta,\xi \in G;$$

$\deg_G a = \theta$ if $a \in A_\theta$.

Examples: $G=\mathbb{Z}_2$

$$\mathbf{1}.\ M_{k,l}(F)=\left\{ \begin{array}{c} \\ k \\ l \end{array} \quad \begin{array}{c} \quad k \qquad l \\ \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)_0 \end{array} \oplus \left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right)_1 \right\}.$$

$$\mathbf{2}.\ M_n(F)\otimes_F F\mathbb{Z}_2=(M_n(F))_0\oplus(M_n(F)\cdot c)_1, \quad c^2=1.$$

1, 2 are all simple finite dimensional $\mathbb{Z}_2$-graded algebras.

$$I \trianglelefteq A \quad \text{is graded if} \quad A_\theta\cap I=I_\theta\subseteq I, \quad \forall\theta\in G.$$

$$\varphi: A\to B \quad \text{is graded if} \quad \varphi(A_\theta)\subseteq B_\theta \quad \forall\theta\in G.$$

Simplectic involution on $M_{k,k}(F)$, transpose involution on $M_{k,l}(F)$ are graded.

We will consider G**-graded algebras** $A=\oplus_{\theta\in G}A_\theta$ **with graded involution** $*$.

$a^*=a$ symmetric element , $a^*=-a$ skew-symmetric element .
$A=A^+\oplus A^-$,

$$A^+=\{a\in A|a^*=a\}=\left\{\frac{a+a^2}{2}|a\in A\right\},$$
$$A^-=\{a\in A|a^*=-a\}=\left\{\frac{a-a^2}{2}|a\in A\right\}. \qquad (1)$$

$$* \text{ is graded if} \qquad A=\bigoplus_{\theta\in G}A_\theta^+\oplus\bigoplus_{\theta\in G}A_\theta^-.$$

Free graded algebra with involution.

$\widetilde{X} = \{ \; x_{i\theta}, \;\; x^*_{i\theta} \mid i \in \mathbb{N}, \theta \in G \; \}$ pairvise different,
$F\langle \widetilde{X} \rangle -$ free G-graded algebra with involution.

$$\deg_G \; x^{\delta_1}_{i_1\theta_1} \cdots x^{\delta_s}_{i_s\theta_s} = \theta_1 + \cdots + \theta_s,$$

$$(x^{\delta_1}_{i_1\theta_1} \cdots x^{\delta_s}_{i_s\theta_s})^* = x^{\delta_s+1}_{i_s\theta_s} \cdots x^{\delta_1+1}_{i_1\theta_1};$$

$$x^\delta \in \{x, x^*\}, \qquad x^{\delta+1} = \begin{cases} x^*, & x^\delta = x \\ x, & x^\delta = x^*. \end{cases}$$

Example:

$$(x_{10}x^*_{21}x^*_{31}x_{41})^* = x^*_{41}x_{31}x_{21}x^*_{10} \quad (G = \mathbb{Z}_2).$$

$Y = \{ \; y_{i\theta} \mid i \in \mathbb{N}, \theta \in G \; \} -$ symmetric variables,
$Z = \{ \; z_{i\theta} \mid i \in \mathbb{N}, \theta \in G \; \} -$ skew-symmetric variables.
$F\langle Y, Z \rangle -$ free G-graded algebra with involution,
$y^*_{i\theta} = y_{i\theta}, \qquad z^*_{i\theta} = -z_{i\theta}.$

$$F\langle Y, Z \rangle \cong F\langle X^* \rangle \qquad \text{by}$$

$$y_{i\theta} = \frac{x_{i\theta} + x^*_{i\theta}}{2}, \quad z_{i\theta} = \frac{x_{i\theta} - x^*_{i\theta}}{2};$$
$$x_{i\theta} = y_{i\theta} + z_{i\theta}, \quad x^*_{i\theta} = y_{i\theta} - z_{i\theta}$$

$f = f(x_{1\theta_1}, \ldots x_{n\theta_n}) \in F\langle X^* \rangle, \quad f \neq 0.$
G-graded algebra $A = \bigoplus_{\theta \in G} A_\theta$ with involution $*$
satisfies the graded $*$-identity $f(x_{1\theta_1}, \ldots x_{n\theta_n}) = 0$

$$\text{if} \;\; f(a_1, \ldots a_n) = 0 \quad \forall \; a_i \in A_{\theta_i}.$$

$$Id^{si}(A) = \{ \; f \in F\langle X^* \rangle \mid f = 0 \text{ in } A \; \} \lhd F\langle X^* \rangle.$$

$I = Id^{si}(A)$ is G-graded, $*$-invariant ideal of $F\langle X^*\rangle$, closed under all graded $*$-invariant endomorphisms of $F\langle X^*\rangle$.

$$x_{i\theta} \mapsto g_i, \qquad g_i \in (F\langle X^*\rangle)_\theta.$$

Free algebra with involution.

Without grading.

$$X^* = \{\ \ x_i,\ \ x_i^* \mid i \in \mathbb{N}\}\ \text{ pairwise different,}$$
$$F\langle X^*\rangle -\ \text{ free algebra with involution.}$$

$$(x_{i_1}^{\delta_1}\cdots x_{i_s}^{\delta_s})^* = x_{i_s}^{\delta_s+1}\cdots x_{i_1}^{\delta_1+1};$$
$$x^\delta \in \{x, x^*\}, \qquad x^{\delta+1} = \begin{cases} x^*, & x^\delta = x \\ x, & x^\delta = x^*. \end{cases}$$

$$F\langle Y, Z\rangle \cong F\langle X^*\rangle$$

$Y = \{\ \ y_i \mid i \in \mathbb{N}\ \}$ – symmetric variables,
$Z = \{\ \ z_i \mid i \in \mathbb{N}\ \}$ – skew-symmetric variables.
$F\langle Y, Z\rangle -$ free algebra with involution,
$y_{i\theta}^* = y_{i\theta}, \qquad z_{i\theta}^* = -z_{i\theta}.$

$f = f(x_1, \ldots x_n) \in F\langle X^*\rangle, \quad f \neq 0.$
Algebra A with involution $*$
satisfies the (non graded) $*$-identity $f(x_1, \ldots x_n) = 0$

$$\text{if }\ f(a_1, \ldots a_n) = 0 \quad \forall\, a_i \in A.$$

$$Id^*(A) = \{\ f \in F\langle X^*\rangle \mid f = 0 \text{ in } A\ \} \lhd F\langle X^*\rangle.$$

$I = Id^*(A)$ is $*$-invariant ideal of $F\langle X^*\rangle$, closed under all $*$-invariant endomorphisms of $F\langle X^*\rangle$.

$$x_i \mapsto g_i, \qquad g_i \in F\langle X^*\rangle.$$

$$Id^{si}(A) \begin{array}{l} \supseteq Id^*(A) \ \ *\text{-id.} \\ \supseteq Id^G(A) \ \ \text{gr. id.} \end{array} \supseteq Id(A) \quad \text{p. i.}$$

Specht problem: Is any ideal of identities finitely generated? Is it true that any identity of A follows from some finite family of identities?

Answers for associative algebras, char F =0:

1. For ordinary identities: **yes** (A.R.Kemer).

2. For G-graded identities: **yes** (G is finite abelian group, IS; G is any finite group, E.Aljadeff, A.Belov).

3. For identities with action of a finite abelian group of automorphisms over algebraically closed field: **yes** (automatic consequence of the previous result).

$$\widehat{G} = \{\chi_1, \ldots, \chi_m\} - \text{ irreducible characters of } G, \qquad m = |G|.$$

$$\widehat{G} \cong G.$$

G-grading on A is equivalent to $\widehat{G}$-action on A :

$$a = a_{\theta_1} + \cdots + a_{\theta_m} \in A,$$

$$\chi(a) \ = \ \chi(\theta_1)\, a_{\theta_1} \ + \cdots + \ \chi(\theta_m)\, a_{\theta_m}, \qquad \chi \in \widehat{G}.$$

Theorem 1 (A.Giambruno, S.Mishchenko, M.Zaicev) *Let F be an algebraically closed field of zero characteristic, G be a finite abelian group, A be an associative algebra. Then ideal of G-graded identities of A coincides with ideal of G-identities $Id^G(A) = Id(A|\widehat{G})$.*

PI-representability.

Algebra A is **PI-representable** if it satisfies exactly the same identities as some finite-dimensional algebra.

A is **finitely generated** PI-algebra.

Theorem 2 (A.R.Kemer) *For any f.g. PI-algebra A $\exists$ finite dimensional algebra C such that: $Id(A) = Id(C)$.*

Theorem 3 (E.Aljadeff, A.Belov) *If G is a finite group then for any f.g. G-graded PI-algebra A $\exists$ finite dimensional G-graded algebra C such that: $Id^G(A) = Id^G(C)$.*

A is **finitely generated** PI-algebra with involution.

Theorem 4 (IS) *For any f.g. algebra A with involution $\exists$ finite dimensional algebra C with involution such that: $Id^*(A) = Id^*(C)$.*

Theorem 5 (IS) *For any f.g. $\mathbb{Z}_p$-graded algebra A with involution $\exists$ finite dimensional $\mathbb{Z}_p$-graded algebra with involution C such that: $Id^{si}(A) = Id^{si}(C)$, p is a prime integer or $p = 4$.*

Specht problem:

4. For identities with involution answer is **yes**.

Problems:

1. Are finitely generated **group-graded algebras with involution** PI-representable for any finite group?

2. Is it true that any group-graded algebra with involution has a finite basis of graded identites with involution? (for any finite group)

3. Are finitely generated **algebras with action of a finite group of automorphisms** PI-representable?

4. Is it true that any algebra has a finite basis of identites with automorphisms? (for action of any finite group)

5. Are finitely generated **algebras with action of a finite group of automorphisms and anti-automorphisms** PI-representable?

6. Is it true that any algebra has a finite basis of identites with automorphisms and anti-automorphisms? (for action of any finite group)

THANK YOU!