Groups, Group rings and the Yang-Baxter equation

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Motivation

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- Study of finitely presented algebras defined by homogeneous relations
- Study of (semi)group algebras
- Construction of algebras, monoids, with "nice" arithmetical structure
- Examples showing up in other areas, e.g. Yang-Baxter equation

In this talk: report on joint work with F. Cedo, J. Okninski and A. del Rio.

There is another recent approach by Ben David and Ginosar using cohomology (work in progress).

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Set-theoretic solutions

Set-theoretic solutions

V a finite dimensional vector space, with basis X
$R : V \otimes V \to V \otimes V$, a bijective linear map
$R_{ij} : V \otimes V \otimes V \to V \otimes V \otimes V$, R acting on (i, j)-component

PROBLEM
Find all solutions R of the quantum Yang-Baxter equation

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12}.$$

PROBLEM: Drinfeld 1992
Find all solutions induced by a linear extension of

$$\mathcal{R} : X \times X \to X \times X.$$

$$\tau : X \times X \to X \times X : (x, y) \mapsto (y, x)$$

$\mathcal{R}$ is a set theoretic solution $\Leftrightarrow$ $r = \tau \circ \mathcal{R}$ is a solution of the braided equation $r_{12}\, r_{23}\, r_{12} = r_{23}\, r_{12}\, r_{23}$

We write $r : X \times X \to X \times X : (x, y) \mapsto (\sigma_x(y),\ \gamma_y(x))$. Such (X, r) (or r) is called a set-theoretic solution of the Yang-Baxter equation.

r is involutive if $r^2 = id$.

A map r is left (right) non-degenerate if each γ_y (respectively σ_x) is bijective.

If X is finite then left=right non-degenerate for involutive set-theoretic solutions [JO].

Group Interpretation

Theorem (GV)

$|X| = n < \infty$ and $r : X \times X \to X \times X$.
If r is a non-degenerate involutive set-theoretic solution then for every $f \in \mathrm{Sym}_n$ there exists a bijection

$$v : \mathrm{FaM}_n = \langle u_1, \ldots, u_n \rangle \to S$$

where

$$S = \langle x_1, \ldots, x_n \mid x_i x_j = x_k x_l \text{ if } r(x_i, x_j) = (x_k, x_l) \rangle,$$

such that $v(1) = 1$, $v(u_i) = x_{f(i)}$ and

$$\{v(u_1 a), \ldots, v(u_n a)\} = \{x_1 v(a), \ldots, x_n v(a)\}$$

for all $a \in \mathrm{FaM}_n$.
And conversely.

Such an S is called a monoid of I-type. It has a group of fractions $G(X,r)$ called a group of I-type (or structure group).

$$G(X,r) = \langle x_1, \ldots, x_n \mid x_i x_j = x_k x_l \text{ if } r(x_i, x_j) = (x_k, x_l) \rangle.$$

Theorem (ESS: for groups)

A monoid (resp. group) S is of I-type if and only if

$$S \cong \{(a, \sigma_a) \mid a \in \mathrm{FaM}_n\} \subseteq \mathrm{FaM}_n \rtimes \mathrm{Sym}_n$$

(resp. $\subseteq \mathrm{Fa}_n \rtimes \mathrm{Sym}_n$) with $\sigma : \mathrm{Fa}_n \to \mathrm{Sym}_n$.

$G(X, r) = S\{z^m \mid m \in \mathbb{Z}\}$, *with* $z = (u, \sigma_u)^{|\sigma_u|}$, *where* $u = u_1 \cdots u_n$.

$K = \{(a, 1) \mid a \in \mathrm{Fa}_n, \sigma_a = 1\}$ *is a free abelian subgroup that is normal and of finite index.*

$G(X, r)/K \cong \{\sigma_a \mid a \in \mathrm{Fa}_n\} = \langle \sigma_{u_i} \mid 1 \leq i \leq n \rangle$.
notation: $\mathcal{G}(X, r)$, called involutive Yang-Baxter group(IYB).

Properties of groups of I-type

A group $\mathcal{G}(X, r)$ of I-type has the following properties:

- abelian-by-finite
- torsion-free
- solvable [R]

The group algebra $K[G(X, r)]$ has nice arithmetical properties:

- a domain
- noetherian, P.I., maximal order

Proposition

(CJO) (useful for verifying solution)
Let X be a finite set and
$r : X \times X \to X \times X : (x, y) \mapsto (\sigma_x(y), \gamma_y(x))$.

Then, (X, r) is a right non-degenerate involutive set-theoretic solution of the Yang-Baxter equation if and only if

1. $r^2 = id_{X^2}$,
2. $\sigma_x \in \mathrm{Sym}_X$, *for all* $x \in X$,
3. $\sigma_x \circ \sigma_{\sigma_x^{-1}(y)} = \sigma_y \circ \sigma_{\sigma_y^{-1}(x)}$, *for all* $x, y \in X$.

Approach 1
to determine set-theoretic solutions and Problems

Problem 1: Characterize groups of I-type.

Problem 1a: Classify involutive Yang-Baxter groups.

Problem 1b: Describe all groups of I-type that have a fixed associated IYB group.

Theorem

(CJR)

- *If G is IYB then its Hall subgroups are IYB.*
- *The class of IYB groups is closed under direct products.*
- *$A \rtimes H$ is IYB if A is finite abelian and H is IYB.*
- *If G is IYB and H is an IYB subgroup of Sym_n then the wreath product of G and H is IYB.*
- *Any finite solvable group is isomorphic to a subgroup of an IYB.*
- *the Sylow subgroups of Sym_n are IYB.*
- *Any finite nilpotent group is isomorphic to a subgroup of an IYB group.*
- *Every finite nilpotent group of class 2 is IYB.*

Problem 2: Are finite solvable groups IYB?

Decomposability and Multipermutation Solutions: approach 2

Decomposability and Multipermutation Solutions

Theorem

(ESS) If $G(X,r)$ is a group of I-type then

$$G(X,r) = G_{(1)} \cdots G_{(m)}$$

with

$$G_{(i)} = \{(a, \sigma_a) \mid a \in \langle u_j \mid u_j \in C_i\}$$

where

$$C_i = \{\sigma_a(u_i) \mid a \in \mathrm{Fa}_n\}$$

and

$$G_{(i)} G_{(j)} = G_{(j)} G_{(i)}.$$

(R): If G is square free then $m > 1$, i.e. $G(X,r)$ is decomposable.
not true if not square free.

Multipermutation Solutions

Let (X, r) be a non-degenerate involutive set-theoretic solution of the Yang-Baxter equation.
$\sim$ equivalence relation on X defined by

$$x \sim y \Leftrightarrow \sigma_x = \sigma_y.$$

Induced solution

$$\mathrm{Ret}(X, r) = (X/\sim, \tilde{r})$$

with

$$\tilde{r}([x], [y]) = ([\sigma_x(y)], [\gamma_y(x)]),$$

where $[x]$ denotes the $\sim$-class of $x \in X$.
Smallest m nonnegative integer so that $|\mathrm{Ret}^m(X, r)| = 1$ is called a multipermutation solution of level m; if it exists (solution is retractable).

If X is finite and multipermutation solution then $G(X,r)$ is a poly-(infinite cyclic). (and thus $KG(X,r)$ is obviously a domain. What about the Kaplansky conjecture?)

Exist examples of groups of I-type that are NOT poly-(infinite cyclic) and thus not a multipermutation solution.

(JO) $G = \langle x_1, x_2, x_3, x_4 \mid x_1x_2 = x_3x_3,\ x_2x_1 = x_4x_4,$

$$x_1x_3 = x_2x_4,\ x_1x_4 = x_4x_2,\ x_2x_3 = x_3x_1,\ x_3x_2 = x_4x_1\rangle$$

is of I-type with $\mathcal{G}(X,r) = D_8$.
Contains $\langle x, y \mid x^{-1}y^2x = y^{-2},\ y^{-1}x^2y = x^{-2}\rangle$ and it is not poly-infinite cyclic (not u.p. group).

Problems

Problem 3 (Gateva-Ivanova):

Every set-theoretic non-degenerate involutive square-free solution (X, r) of the Yang-Baxter equation of cardinality $n \geq 2$ is a multipermutation solution of level $m < n$.

Problem 4 (Gateva-Ivanova and Cameron):

Let (X, r) be a finite multipermutation square-free solution of the Yang-Baxter equation with $|X| > 1$ and multipermuation level m.

1. Can we find a lower bound for the solvable length of the group of I-type associated to (X, r) in terms of m?
2. Are there multipermutation square-free solutions (X, r) of arbitrarily high multipermutation level with an abelian IYB group $\mathcal{G}(X, r)$?

Theorem

(CJO) Let (X, r) be a finite non-degenerate involutive set-theoretic solution of the Yang-Baxter equation. If its associated IYB group $\mathcal{G}(X, r)$ is abelian, then (X, r) is a multipermutation solution.

Corollary

(CJO) Let (X, r) be a finite non-degenerate involutive set-theoretic square-free solution of the Yang-Baxter equation. If its associated IYB group $\mathcal{G}(X, r)$ is abelian, then (X, r) is a strong multipermutation solution, i.e. there exist $\sigma_x = \sigma_y$ for some x and y in the same $\mathcal{G}(X, r)$-orbit.

Theorem

(CJO) Let n be a positive integer. Then there exists a finite multipermutation square-free solution of the Yang-Baxter equation of multipermutation level n such that its associated IYB group is an elementary abelian 2-group. ((R): non-square free examples)

Braces and the Yang-Baxter equation: approach 3

Definition

(R) A right brace is a set G with two operations $+$ and $\cdot$ such that $(G,+)$ is an abelian group, $(G,\cdot)$ is a group and

$$(a+b)c+c=ac+bc,$$

for all $a,b\in G$.
Such a G is a two-sided brace if it is also a left brace, i.e.

$$a(b+c)+a=ab+ac,$$

for all $a,b,c\in G$.

Proposition

(R) If $(G, +, \cdot)$ is a two-sided brace then $(G, +, *)$ is a radical ring (with $a * b = ab - a - b$).
Conversely, if $(R, +, \cdot)$ is a radical ring then $(R, +, \circ)$ is a two-sided brace (with $a \circ b = ab + a + b$).

Note that the multiplicative identity 1 of $(G, \cdot)$ is the same as the additive identity 0 of $(G, +)$.

For $a \in G$ let $\lambda_a, \rho_a \in \mathrm{Sym}_G$, such that

$$\rho_a(b) = ba - a \text{ and } \lambda_a(b) = ab - a.$$

If G is a left brace then λ_a is an automorphism of $(G, +)$, and $\lambda_{ab} = \lambda_a \lambda_b$.

Lemma

(R) Let G be a left brace. The following properties hold.

(i) $a\lambda_a^{-1}(b) = b\lambda_b^{-1}(a)$.

(ii) $\lambda_a\lambda_{\lambda_a^{-1}(b)} = \lambda_b\lambda_{\lambda_b^{-1}(a)}$.

(iii) *The map $r\colon G \times G \longrightarrow G \times G$ defined by $r(x,y) = (\lambda_x(y), \lambda^{-1}_{\lambda_x(y)}(x))$ is a non-degenerate involutive set-theoretic solution of the Yang-Baxter equation.*

The set-theoretic solution of the Yang-Baxter equation (G, r) is called the solution of the Yang-Baxter equation associated to the left brace G.

Proposition

(CJO, CJR) A group G is the multiplicative group of a left brace if and only if there exists a group homomorphism $\mu : G \longrightarrow \mathrm{Sym}_G$ such that $x\mu(x)^{-1}(y) = y\mu(y)^{-1}(x)$ for all $x, y \in G$.

Corollary

(CJO, CJR) A finite group G is an IYB group if and only if it is the multiplicative group of a finite left brace.

Theorem

Let $(A,+)$ be an abelian group. Let

$$\mathcal{B}(A)=\{(A,+,\cdot)\mid (A,+,\cdot) \text{ is a left brace}\}$$

and

$$\mathcal{S}(A)=\{G\mid G \text{ is a subgroup of } A\rtimes \mathrm{Aut}(A)$$
$$\text{of the form } G=\{(a,\phi(a))\mid a\in A\}\}.$$

The map $f:\mathcal{B}(A)\to\mathcal{S}(A)$ defined by

$$f(A,+,\cdot)=\{(a,\lambda_a)\mid a\in A\}$$

is bijective.

Proposition

(CJO) A group G is of I-type if and only if it is isomorphic to the multiplicative group of a left brace B such that the additive group of B is a free abelian group with a finite basis X such that $\lambda_x(y) \in X$ for all $x, y \in X$.

Braces, groups rings and the Yang-Baxter equation: approach 4

Proposition

(S) Let G be a group. Then G is the multiplicative group of a left brace if and only if there exists a left ideal L of $\mathbb{Z}[G]$ such that

(i) *the augmentation ideal $\omega(\mathbb{Z}[G]) = G - 1 + L$ and*

(ii) $G \cap (1 + L) = \{1\}$.

Proposition

Let G be a group. Then G is the multiplicative group of a two-sided brace if and only if there exists an ideal L of $\mathbb{Z}[G]$ such that

(i) *the augmentation ideal $\omega(\mathbb{Z}[G]) = G - 1 + L$ and*

(ii) $G \cap (1 + L) = \{1\}$.

Has implications for the integral isomorphism problem. It follows

$$U(\mathbb{Z}G) = (\pm G)H \text{ and } \pm G \cap H = \{1\}$$

with $H = (1 + L) \cap U(\mathbb{Z}G)$.

If L is a two-sided ideal, then H is a normal subgroup and this is a normal complement in $U(\mathbb{Z}G)$ of $\pm G$.

If G is a finite nilpotent group, then a positive answer to existence of a normal complement gives a positive answer for the integral group ring isomorphism problem, i.e. if $\mathbb{Z}G \cong \mathbb{Z}G_1$ then $G \cong G_1$.

Positive answer for G of class two. In general it is an open problem (although ISO has a positive answer for nilpotent groups).

The counter example of Hertweck to ISO maybe indicates that a positive answer to complements is maybe not true in general.

Construction of Braces

- Abelian groups
- Nilpotent groups of class 2 (Ault and Watters): they are the adjoint group of a radical ring
- Nilpotent groups of class 2 are the adjoint group of a radical ring of nilpotency class 3 in case
  - $G/Z(G)$ is the weak direct product of cyclic groups
  - $G/Z(G)$ is a torsion group
  - Every element of G' has a unique square root.
- Hales and Passi: previous not always true, but true if $G/Z(G)$ is uniquely 2-divisible, or if G/N is torsion-free and a weak direct product of rank one groups for some normal subgroup N such that $G' \subseteq N \subseteq Z(G)$. - also true for $H = \{g^2 z \mid g \in G,\ z \in Z(G)\}$ and associated solution of the Yang-Baxter equation to the brace H is square free.

- Exist nilpotent class 2 groups which admit a structure of a left brace that is not a right brace
- open problem: does any finite nilpotent group admit a structure of a left brace? (i.e. are they IYB groups?) They do not necessarily admit a two-sided brace as not all such groups are adjoint groups of radical rings.