Surface braids and mapping class group IV
(Pure) Braid groups on surfaces of positive genus:
other definitions and topological tools

Paolo Bellingeri

Laboratoire de Mathématiques Nicolas Oresme, Université de Caen

Atlantic Algebra Centre
Let $\Sigma$ be an oriented surface, $\mathcal{P} = \{p_1, \ldots, p_n\} \subset \Sigma$

**Mapping class group of $\Sigma$**:
$$\mathcal{M}(\Sigma) = \left\{ h : \Sigma^+ \to \Sigma^+ ; h|_{\partial \Sigma} = \text{Id} \right\} / \sim$$

**$n$-punctured Mapping class group of $\Sigma$**:
$$\mathcal{M}_n(\Sigma) = \left\{ \begin{array}{c} h : \Sigma^+ \to \Sigma^+ \\ h(p_i) \in \mathcal{P} \ i = 1, \ldots, n \\ h|_{\partial \Sigma} = \text{Id} \end{array} \right\} / \sim$$

Let $\psi_n : \mathcal{M}_n(\Sigma) \to \mathcal{M}(\Sigma)$ be the *forgetting map*.

**Theorem (Birman).**

1. $\to \mathcal{B}_n(\Sigma) / \mathcal{ZB}_n(\Sigma) \to \mathcal{M}_n(\Sigma) \to \mathcal{M}(\Sigma) \to 1$ if $\Sigma = S^2$ or $T^2$.
2. $\to \mathcal{B}_n(\Sigma) \to \mathcal{M}_n(\Sigma) \to \mathcal{M}(\Sigma) \to 1$ otherwise.
Applications: automorphisms of surface braids groups

Extended Mapping class group of $\Sigma$:
$$\mathcal{M}^*(\Sigma) = \{ h : \Sigma \to \Sigma; h|_{\partial \Sigma} = Id \} / \sim$$

Theorem (B. 2007). Let $\Sigma_g$ be a closed oriented surface of genus $g > 1$ and $n > 2$.

- The group $\text{Aut}(B_n(\Sigma))$ is isomorphic to $\mathcal{M}_n^*(\Sigma_g)$.
- The group $\text{Out}(B_n(\Sigma))$ is isomorphic to $\mathcal{M}^*(\Sigma_g)$.

This result has been extended to other orientable surfaces by several authors (An, 2017).

Proposition. Let $g \geq 0$. $B_n(\Sigma_g)$ is hopfian.

Theorem (Kida-Yamagata. 2011). Let $g > 1$: any isomorphism between finite index subgroups of $B_n(\Sigma_g)$ is induced by an extended mapping class.

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A group presentation for $P_n(\Sigma_g)$.

Recall that Fadell-Neuwirth fibration

\[ p : \mathbb{F}_{n+1} \Sigma \to \mathbb{F}_n \Sigma, \quad p((x_1, \ldots x_n, x_{n+1})) = (x_1, \ldots x_n) \]

implies that:

\[
(PBS) \quad 1 \to \pi_1(\Sigma \setminus \{n \text{ points}\}) \to P_{n+1}(\Sigma) \xrightarrow{\pi_{n+1}} P_n(\Sigma) \to 1;
\]

Using (PBS) and Lindon-Schupp method we can obtain group presentations for $P_m(\Sigma)$. 
The group $P_n(\Sigma_{g,1})$ admits the following presentation:

- **Generators**: $\{A_{i,j} \mid 1 \leq i \leq 2g + n - 1, 2g + 1 \leq j \leq 2g + n, i < j\}$.

Only non trivial strands of $A_{i,j}$ are presented.
The group $P_n(\Sigma_g, 1)$ admits the following presentation:

- **Generators:** $\{A_{i,j} \mid 1 \leq i \leq 2g + n - 1, 2g + 1 \leq j \leq 2g + n, i < j\}$.

- **Relations:**

  (PR1) $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{r,s}$ if $(i < j < r < s)$ or $(r + 1 < i < j < s)$,
  or $(i = r + 1 < j < s$ for even $r < 2g$ or $r > 2g$);

  (PR2) $A_{i,j}^{-1} A_{i,s} A_{j,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,j,s}^{-1}$ if $(i < j < s)$;

  (PR3) $A_{i,j}^{-1} A_{i,s} A_{j,s} A_{i,j,s}^{-1} A_{i,s}^{-1}$ if $(i < j < s)$;

  (PR4) $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1} A_{j,s}^{-1} A_{r,s} A_{j,s} A_{i,s} A_{j,s} A_{i,s}$
  if $(i + 1 < r < j < s)$ or
  $(i + 1 = r < j < s$ for odd $r < 2g$ or $r > 2g$);

  (ER1) $A_{r+1,j}^{-1} A_{r,s} A_{r+1,j} = A_{r,s} A_{r+1,s} A_{j,s}^{-1} A_{r+1,s}^{-1}$
  if $r$ odd and $r < 2g$;

  (ER2) $A_{r-1,j}^{-1} A_{r,s} A_{r-1,j} = A_{r-1,s} A_{j,s} A_{r-1,s} A_{r,s} A_{j,s} A_{r-1,s}$
  if $r$ even and $r < 2g$.
The sequence

\[ 1 \to P_m(\mathbb{D}^2 \setminus \{n \text{ points}\}) \to P_{n+m} \xrightarrow{\pi_{n,m}} P_n \to 1 \]

splits (i.e. there exists a section) and \( P_{n+m}(\Sigma) \) in an \textit{almost} direct product of \( P_m(\mathbb{D}^2 \setminus \{1 \text{ points}\}) \) and \( P_n \).

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splits for \( n = 1 \) and \( P_{1+m}(\Sigma) \) in an \textit{almost} direct product of \( P_m(\Sigma \setminus \{1 \text{ points}\}) \) and \( P_1(\Sigma) \).
Back to Fadell-Neuwirth

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splits for \( n = 1 \) and \( P_{1+m}(\Sigma) \) in an almost direct product of \( P_m(\Sigma \setminus \{1 \text{ points}\}) \) and \( P_1(\Sigma) \).

BUT:

**Theorem (Gonçalves-Guaschi 2004).** Let \( \Sigma_g \) be a closed surface of genus \( g \geq 2 \). The sequence \((PBS)\) does not split when \( n > 1 \).
The sequence

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splits for $n = 1$ and $P_{1+m}(\Sigma)$ in an *almost* direct product of $P_m(\Sigma \setminus \{1 \text{ points}\})$ and $P_1(\Sigma)$.

**BUT :**

**Theorem (Gonçalves-Guaschi 2004).** Let $\Sigma_g$ be a closed surface of genus $g \geq 2$. The sequence $(PBS)$ does not split when $n > 1$. 
Similarly, given the sequence:

\[(MBS)\quad 1 \to B_m(\Sigma \setminus \{n \text{ points}\}) \to B_{n,m}(\Sigma) \xrightarrow{\pi_{n,m}} B_n(\Sigma) \to 1\]

we have that

**Proposition (B-Godelle-Guaschi 2017)** Let $\Sigma_g$ of genus $g > 1$; then the sequence $(MBS)$ does not split when $n > 1$. 
Let $\Sigma$ be a surface with boundary.

$$(PBS) \quad 1 \to P_m(\Sigma \setminus \{n \text{ points}\}) \to P_{n+m}(\Sigma) \xrightarrow{\pi_{n,m}} P_n(\Sigma) \to 1$$
Let $\Sigma$ be a surface with boundary.

$$(PBS) \quad 1 \to P_m(\Sigma \setminus \{n \text{ points}\}) \to P_{n+m}(\Sigma) \xrightarrow{\pi_{n,m}} P_n(\Sigma) \to 1$$

**Good point:** If $\Sigma$ has boundary components the "natural" section is well defined. (we can see elements of $P_n(\Sigma)$ as elements of $P_{n+m}(\Sigma)$ adding $m$ vertical strands "at the infinity").
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**Bad point:** If $\Sigma$ has boundary components (and genus $g > 0$) there is no section defining an action of $P_n(\Sigma)$ on $P_m(\Sigma \setminus \{n \text{ points}\})$ which is trivial on the abelianization.
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**Bad point:** If $\Sigma$ has boundary components (and genus $g > 0$) there is no section defining an action of $P_n(\Sigma)$ on $P_m(\Sigma \setminus \{n \text{ points}\})$ which is trivial on the abelianization.

**Theorem (B.-Gervais-Guaschi 2008 ; B.-Bardakov 2009)**

Let $\Sigma \neq S^2$ be an oriented surface (possibly with boundary). $P_n(\Sigma)$ is residually torsion-free nilpotent.
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"Proof" : case with one boundary component :

$$1 \to \mathbb{Z}^n \to \mathcal{M}(\Sigma_{g,n+p}) \to \mathcal{P}\mathcal{M}_n(\Sigma_{g,p}) \to 1$$

There is no section (homology !) but the restriction to

$$P_n(\Sigma_{g,1}) \subset \mathcal{P}\mathcal{M}_n(\Sigma_{g,1})$$

has a section. Finally ; gluing holed torus on boundary disks we realize

$P_n(\Sigma_{g,1})$ as a subgroup of $\mathcal{M}(\Sigma_{g+n,1})$ and moreover

$P_n(\Sigma_{g,1}) \subset \mathcal{T}(\Sigma_{g+n,1})$, the Torelli group of $\Sigma_{g+n,1}$. 
$P_n(\Sigma)$ is residually torsion-free nilpotent

"Proof" : closed case :

Recall that

$$(PBS) \quad 1 \rightarrow P_m(\Sigma \setminus \{1 \text{ points}\}) \rightarrow P_{m+1}(\Sigma) \xrightarrow{\pi_{m,1}} P_1(\Sigma) \rightarrow 1$$

The sequence splits and defining a possible section (Guaschi-Gonçalves 2004, B-Bardakov 2009, Gonzáles Meneses-Silvero 2018...) one can verify that $P_{m+1}(\Sigma)$ is an almost direct product of RTFN groups ; then it is also RTFN.
An extension for closed surfaces

\textbf{nth framed pure braid group of } \Sigma_g (B.-Gervais)

\[ FP_n(\Sigma_g) = \ker \Phi_n, \quad \Phi_n : \mathcal{M}(\Sigma_{g,n}) \to \mathcal{M}(\Sigma_g) \]

- Let \( U\Sigma_g \) the unitary tangent space of \( \Sigma_g \) and \( \pi : U\Sigma_g \to \Sigma_g \) the natural projection. Let \( T_n\Sigma_g = (\pi^n)^{-1}(F_n\Sigma_g) \). \( T_n\Sigma_g \) is a classifying space and \( \pi_1(T_n\Sigma_g) \cong FP_n(\Sigma_g) \). Therefore we can define:

\textbf{nth framed braid group of } \Sigma_g : \text{FB}_n(\Sigma_g) = \pi_1(T_n\Sigma_g/ S_n)

- \( FP_1(\Sigma_g) \cong \pi_1(U\Sigma_g) \);

- \( \text{FB}_n(\mathbb{D}^2) \) is isomorphic to the framed braid group \( \mathcal{F}_n := \mathbb{Z}^n \rtimes B_n \) (Ko-Smolinski 1992);

- \( \text{FB}_n(\Sigma_g) \) can be characterised in terms of \textit{framed} mapping class groups.
- $FP_n(\Sigma_g)$ can be seen as a subgroup of $P_{2n}(\Sigma_g)$

![Diagram](image1.png)

- $FB_n(\Sigma_g)$ can be seen as a particular subgroup of $B_{2n}(\Sigma_g)$.

![Diagram](image2.png)
Theorem (B.-Gervais 2012).

1) $FP_n(\Sigma)$ is a central extension of $P_n(\Sigma)$ by $\mathbb{Z}^n$:

\[
\begin{array}{c}
1 
\rightarrow \mathbb{Z}^n 
\rightarrow FP_n(\Sigma) 
\rightarrow P_n(\Sigma) 
\rightarrow 1
\end{array}
\]

(\rightarrow \text{group presentation for } FP_n(\Sigma))

2) (\star) splits if and only if $\Sigma$ has boundary or $\Sigma = \mathbb{T}^2$ : in particular, if $\Sigma$ has boundary, $FP_n(\Sigma) \simeq \mathbb{Z}^n \times P_n(\Sigma)$.

3) The inclusion of $\Sigma_{g,n}$ into $\Sigma_{g,n+m}$ induces a section for the exact sequence induced by the "framed" Fadell-Neuwirth fibration

$p : \mathbb{T}_{n+m}\Sigma_g \rightarrow \mathbb{T}_n\Sigma_g :$

\[
\begin{array}{c}
1 
\rightarrow FP_m(\Sigma_{g,n}) 
\rightarrow FP_{n+m}(\Sigma_g) 
\rightarrow FP_n(\Sigma_g) 
\rightarrow 1
\end{array}
\]
The sequence

\[ 1 \rightarrow FP_m(\Sigma_{g,n}) \rightarrow FP_{n+m}(\Sigma_g) \rightarrow FP_n(\Sigma_g) \rightarrow 1 \]

has the same behavior than the (PBS) sequence in the case of surfaces with boundary.
Nevertheless :

**Proposition (B.-Gervais)**

Let \( \Sigma \neq S^2 \) be an oriented surface (possibly with boundary). \( FP_n(\Sigma) \) is residually torsion-free nilpotent.